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THE FEBRUARY MEETING IN NEW YORK

The two hundred fifty-ninth regular meeting of the American Mathematical Society was held at Columbia University, on Saturday, February 25, 1928, extending through the usual morning and afternoon sessions. Attendance included the following eighty-six members of the Society:

R. C. Archibald, R. G. Archibald, J. W. Arnold, Beetle, Benton, Bratton, Carrie, E. T. Carroll, Alonzo Church, Clark, Clutz, Cramlet, Dehn, Derby, Doermann, Dresden, Engstrom, Feld, Fiske, Fite, D. A. Flanders, Fort, Frink, Gale, Gehman, Gill, Glenn, Gronwall, C. C. Grove, L. S. Hill, Hille, Himwich, Hofmann, Hollcroft, Joffe, M. I. Johnson, R. A. Johnson, Kasner, Kenny, Kline, Knebelman, Koopman, J. P. Kormes, Mark Kormes, Langman, Lefschetz, Lehr, Littauer, L. T. Moore, T. W. Moore, Moyle, Mullins, F. H. Murray, C. A. Nelson, Ore, Pepper, Pfeiffer, Pixley, Post, Raudenbush, Rawles, Reddick, M. S. Rees, Robertson, Roos, Rutt, Schoonmaker, Seely, Serghiesco, Simons, Smail, Virgil Snyder, J. M. Thomas, T. Y. Thomas, Tracey, Veblen, Wedderburn, Weida, Weisner, H. S. White, Whittemore, W. M. Whyburn, Widder, W. A. Wilson, J. W. Young, Margaret M. Young.

There was no meeting of the Council or of the Trustees.

President Snyder presided at the morning session, relieved in the afternoon by Vice-President Young.

Titles and abstracts of the papers read at this meeting follow below. The papers of Gergen, Ghosh, Prasad, Trjitzinsky, G. T. Whyburn, and Zarycki were read by title. Mr. Ghosh was introduced by the Secretary, and Dr. Zarycki by Professor Dunham Jackson.

1. Dr. Louis Weisner: *Polynomials $f[\phi(x)]$ reducible in fields in which $f(x)$ is irreducible.*

Given a polynomial $f(x)$ with coefficients in R and irreducible in R, the author determines all polynomials $\phi(x)$ with coefficients in R such that $f[\phi(x)]$ is reducible in R.

2. Dr. R. G. Archibald: *The impossibility of a separation of types of linear odd divisors of binary quadratic forms.*

This paper develops theorems on the existence of solutions of quadratic congruences in two unknowns and utilizes these results to prove the

impossibility of a separation of types of linear odd divisors of binary quadratic forms. Tables are given in Legendre's *Théorie des Nombres* of the quadratic divisors and of the linear odd divisors of a form t^2+au^2 for certain integral values of a. In many cases the forms of the linear odd divisors are paired off in sets with a single quadratic divisor. This paper is concerned with the remaining cases in Legendre's Tables IV–VII in which the forms of the linear odd divisors of t^2+au^2 are paired off in sets with more than one quadratic divisor. If the linear odd divisor $4ax+\alpha$ corresponds to two or more quadratic divisors of t^2+au^2, the question is raised whether it is possible to find a positive integer n such that all the divisors of t^2+au^2 contained in each of $4any+(j4a+\alpha)$, $j=0, 1, \cdots, n-1$, are represented, respectively, by one and only one of the quadratic divisors of t^2+au^2. By the method here developed, it is found that such a separation is impossible.

3. Professor J. I. Tracey and Dr. L. T. Moore: *Covariant conditions for multiple roots of a binary form.*

This paper gives a method whereby a covariant may be formed which vanishes when an n-ic has a p-fold root, a p_1-fold root, a p_2-fold root, $\cdots$, a p_j-fold root simultaneously. It also enables one to determine the other systems of equality among the roots for which the same covariant will vanish.

4. Professor O. E. Glenn: *On the generalization of the galoisian complex domain.*

For every integral modulus n there exists a form $f(x)=x^2-x+r$, with r integral, having no integral roots modulo n. Assume an imaginary ι such that $f(\iota)\equiv 0 \pmod{n}$. Any polynomial in ι with integral coefficients is reducible by $\iota^2=\iota-r$ to $\alpha\iota+\beta$, where α, β are integral residues of n. These n^2 complex numbers form a domain Δ in which a congruence whose coefficients are residues of n, $F(x)=a_0x^m+a_1x^{m-1}+\cdots+a_m\equiv 0 \pmod{n}$, may have N roots and, if n is composite, $N<$, $=$, or $>m$. This paper, in its present form, is concerned with necessary and sufficient matrix conditions, analogous to the algebraic theory of Stuyvaert and Dines, in order that two polynomials F may have a common root in Δ.

5. Dr. Stefan Serghiesco: *On the number of roots of a system of n equations.*

In a previous paper presented to the Society, the author introduced an integral containing certain differential invariants formed from a given system of n equations, and showed that, in a special case, this integral gives the number of common roots of the system of equations. This present paper contains a further extension of this theory, and shows that by means of the same integral we may obtain a formula giving the general solution of the problem. The formula, which is a difference of two integrals, and the method are entirely different from those used in the Picard-Kronecker theory of the same problem.

6. Dr. W. M. Whyburn (National Research Fellow): *On self-adjoint, ordinary differential equations of the fourth order.*

The general, real, self-adjoint, ordinary differential equation of the fourth order and the second-order equation with coefficients that are complex functions of a real variable are studied through a consideration of the pair of real, second-order differential equations

$$\frac{d^2y}{dx^2}+p(x)y=q(x)z,\ \frac{d^2z}{dx^2}+p(x)z=r(x).y$$

Simple conditions are derived under which the integral curve in the yz-plane will be essentially spiriliform and will make an arbitrarily large number of circulations about the origin as x ranges over the entire real axis. Related integral equations are studied, and used to develop comparison theorems of the type obtained by Birkhoff (Annals of Mathematics, (2), vol. 12, pp. 103–127) for third-order systems. Several other properties of the system are exhibited through their relation to similar properties of the second-order equation $u''+p(x)u=0$. A dynamical interpretation is used for the equations and the hypotheses employed.

7. Professor Tomlinson Fort: *A series resembling factorial series.*

The author considers series of the type

$$\Sigma_{n=1}^{\infty}c_n\lambda_1\lambda_2\cdots\lambda_n/((z+\lambda_1)(z+\lambda_2)\cdots(z+\lambda_n)),$$

where $0\leqq\lambda_{n-1}\leqq\lambda_n\rightarrow\infty$. He proves a variety of theorems concerning convergence and summability. The series in question is seen to bear a relation to the factorial series,

$$\Sigma_{n=1}^{\infty}c_n n!/(z(z+1)\cdots(z+n)),$$

resembling that borne by the Dirichlet series, $\Sigma_{n=1}^{\infty}c_n e^{-\lambda_n z}$, to the special Dirichlet series $\Sigma_{n=1}^{\infty}c_n n^{-z}$.

8. Professor D. V. Widder: *The singularities of functions defined by the Stieltjes integral* $\int_0^{\infty}e^{-st}d\alpha(t)$.

Seeking to generalize power series and Dirichlet series by replacing the discrete set of numbers λ_n that appears in such series, $\Sigma_{n=1}^{\infty}a_n e^{-\lambda_n z}$, by a continuous set, the author considers a Stieltjes integral of the form $f(s)=\int_0^{\infty}e^{-st}d\alpha(t)$, where $\alpha(t)$ is a function of bounded variation in every finite interval, $0\leqq t\leqq R$. Functions of this form, in which $\alpha(t)$ has a continuous derivative, were first studied by Laplace, the function $\alpha'(t)$ having been designated as the determining function, $f(s)$ as the generating function. The present paper undertakes, for the first time, the study of the singularities of such functions, as determined by the generating function. The most important result states that if the function $f(s)$ has singularities at points a, and if the function $\phi(s)=\int_0^{\infty}e^{-st}d\beta(t)$ has singularities at points b, then $F(s)=\int_0^{\infty}e^{-st}\alpha(t)d\beta(t)$ has singularities at most at points $a+b$ and b, under certain conditions imposed on the rates of increase of $f(s)$ and $\phi(s)$

on the vertical lines of the complex plane. From further developments it is found that the well known theorems of Hadamard, Hurwitz, and Faber, three of the most fundamental theorems of analytic continuation, appear as special cases under the present theory.

9. Dr. T. H. Rawles: *Two classes of periodic orbits with repelling forces.*

The purpose of this paper is to obtain certain classes of periodic orbits of a system consisting of one body of very great mass which attracts two mutually repellent bodies of very small mass. It will be shown that if certain conditions of symmetry are imposed the orbits of the two small bodies must lie in two parallel planes or else be coplanar. Expressions for these two types will be obtained.

10. Professor W. A. Wilson: *Some properties of upper semi-continuous collections of continua.*

Let $M = \Sigma[X]$ denote the sum of a set of disjoint bounded continua in a plane, let t denote any point of a circumference C, and let there be a (1, 1) correspondence between the elements of M and the points of C such that $X = f(t)$ is upper semi-continuous in C. R. L. Moore has shown that if no element X separates the plane, then M divides the plane into two domains whose frontiers are parts of M. By imposing the condition that $X = f(t)$ is a *minimal* upper semi-continuous function, that is, that there is no function $Y = g(t)$ defined over C such that each Y is a sub-continuum of X and some $Y \neq X$, it is shown that M is the frontier of both its complementary domains. If, in addition, the condition that the elements X do not separate the plane is removed, M is still the common frontier of two complementary domains and the frontier of each other complementary domain is a part of some element. Finally, if $f(t)$ is a minimal upper semi-continuous function defined over a segment $a \leq t \leq b$, and no element separates the elements $X_a = f(a)$ and $X_b = f(b)$, M is a continuum irreducible between the sets X_a and Y_b.

11. Mr. J. J. Gergen: *On accessible points on the boundary of a three-dimensional region.*

The author is engaged in generalizing to three dimensions some of the results of W. F. Osgood and E. H. Taylor established in *Conformal transformations on the boundaries of their regions of definition* (Transactions of this Society, vol. 14 (1913), p. 277). The chief result of the present paper is that an accessible point on the boundary of a simply connected three-dimensional open region, on which the Green's function vanishes, corresponds at most to a point set of zero measure on the unit sphere, when the interior of the region is transformed into the interior of the unit sphere by means of the correspondence set up by the Green's functions and their orthogonal trajectories. Examples can be given to show that this is the most general result that can be obtained merely in these terms,

without restricting the character of the boundary point beyond mere accessibility.

12. Professor Ganesh Prasad: *On the nature of θ in the mean-value theorem of the differential calculus.*

If $f(x)$ is a single-valued function which is finite and continuous in an interval (a, b) the ends being included, then the relation $f(x+h)-f(x)=hf'(x+\theta h)$, $0<\theta<1$, holds for every value of x and h for which the interval $(x, x+h)$ is in the interval (a, b), provided that *either* $f'(x)$ exists for every internal point of (a, b) *or* a certain less restrictive condition is satisfied. The object of the present paper is to consider θ as a function of h, x being taken a constant, say 0, and to show (1) that θ is not necessarily single-valued, (2) that, even if we fix upon a special group of values of θ, which may be called a single-valued function of h, such a function is not necessarily continuous, and (3) that, even if θ is continuous, it is not necessarily differentiable.

13. Mr. Suddhodan Ghosh: *On the solution of the equations of elastic equilibrium suitable for elliptic boundaries.*

A solution in elliptic coordinates of the two-dimensional problem of equilibrium of an elastic isotropic body in the absence of body forces is given in the present paper, and applied to two problems of plane strain. A single-valued solution is found, suitable for the case of an elliptic cylindrical cavity in an infinite solid, and also a many-valued solution applicable to the state of strain in an elliptic cylindrical shell which has suffered dislocation due to a triangular axial fissure. Love has given an outline of a method for solving problems in plane strain in elliptic coordinates, but his method is only applicable to cases where the surface displacements are given; it is shown that the method developed in this paper lends itself easily to the construction of the solution of the particular problem dealt with by Love.

14. Professor W. J. Trjitzinsky: *Functions with assigned initial values.*

The problem of constructing indefinitely differentiable functions of a real variable with assigned initial-values at a point has been undertaken by de la Vallée Poussin in connection with the study of quasi-analytic functions; he gives the representation in the form $\sum\alpha_i \cos m_i x$. In the present paper the problem is treated more completely; representations are given (1) in a series of power series, (2) a series of infinite integrals, and (3) with assigned initial values at a singular point, which is exhibited as a limiting point of a sequence of poles. An asymptotic representation is given in connection with the second method; also, applying a formula of Darboux, a more general asymptotic expansion is derived.

15. Professor G. T. Whyburn: *Concerning the cuttings of a continuous curve.*

Let M be any bounded plane continuous curve, and if R is an open subset of M, let $F(R)$ denote the boundary of R with respect to M. In this paper the following results are proved. (1) There exist in M, an irreducible cutting K of M, and a component H of K such that no point of H is accessible from any component of $M-K$. (2) If R is any connected open subset of M, then each component H of $F_m(R)$ is "accessible" from R, that is, if A is any point of R, there exists a continuum N containing both A and H, which is a subset of $R+H$. (3) If K is an irreducible cutting of M between two points A and B of M, and R_c is any component of $M-K$ which contains neither A nor B, then $F_m(R_c)$ contains points of not more than two components of K. (4) Suppose K is an irreducible cutting of M which itself has at least two components and is such that $M-K$ has at least three components. Then if any component of K is decomposable, K consists of exactly two points.

16. Dr. H. P. Robertson (National Research Fellow): *Invariants of contact transformations.*

A tensor calculus applicable to contact transformations is developed, and by means of it a theory of the invariants of such transformations constructed. The Hamiltonian theory of dynamics is identified with the invariant theory of a single scalar.

17. Professor Einar Hille: *Note on the behavior of certain power series on the circle of convergence with application to a problem of Carleman.*

The paper contains a study of the properties of convergence of power series of the form $\sum f(n) \exp [2i\, a(n)]z^n$, where $f(n)$ and $a(n)$ are subjected to certain restrictions. There exist power series of this type which are continuous in and on the unit circle and are such that $\sum [f(n)]^r$ converges for no $r<2$. This paper will appear in an early issue of the Proceedings of the National Academy.

18. Professor Solomon Lefschetz: *Duality relations in analysis situs.*

Given two complexes C_n and $C_k \subset C_n$, where C_k includes the boundary F of C_n, there may be defined certain simultaneous invariants of the pair which generalize the Betti numbers. These numbers are shown to satisfy duality relations that include directly or indirectly all those now known in analysis situs, notably the relations apparently wholly unrelated due to Poincaré and to Alexander. The same relations are extended to the case of any closed set $G \subset C_n$, with $F \subset G$.

19. Dr. Miron Zarycki: *General properties of Cantorian coherences.*

The author develops on a postulational basis that part of point set theory which concerns coherences and adherences. The postulates are

those underlying the usual algebra of logic and three others which become familiar theorems of point set theory when the undefined terms are given a particular interpretation. These three postulates are shown to form an absolutely independent set.

20. Professor J. R. Kline: *Concerning the dimension of a continuous curve every subcontinuum of which is a continuous curve.*

The author proves that every continuous curve, which has the property that every one of its subcontinua is a continuous curve, must be one-dimensional in the sense of Menger and Urysohn.

21. Professor Edward Kasner: *General theory of element-point transformations.*

The author studies the general transformation T from a lineal element (x, y, y') of a first plane I to a point (X, Y) of a second plane II. To a point of I corresponds a curve of II. To a point of II corresponds a series of elements in I. By eliminating y' from $X=X(x, y, y')$, $Y=Y(x, y, y')$, we obtain a related contact transformation Γ. To a curve in I corresponds by T a curve in II, but to a curve in II corresponds a field of elements and therefore ∞^1 curves in I. Special types of transformation are studied, and in particular the type arising in the author's theory of polygenic functions.

22. Professor Edward Kasner: *Additive groups of transformations.*

The author considers sets of point transformations which form a group in a sense different from the usual sense. Instead of combining two transformations T_1, T_2 by the usual multiplication T_1T_2, he considers vectorial addition. If T_1 transforms the point P into P_1, and T_2 transforms P into P_2, then the sum of the vectors PP_1 and PP_2 gives PP_3 and the transformation from P to P_3 is defined as T_1+T_2. For example, affine transformations form a group in this sense, but projective transformations do not. Attention was called to additive groups incidentally in connection with Darboux transformations by the author in 1909 (see Transactions of this Society, vol. 15, p. 212, last foot note). Many new examples are now given. in particular the set of all harmonic transformations $X=\phi(x, y)$, $Y=\psi(x, y)$, where ϕ and ψ obey the Laplace equation. (See Proceedings of the National Academy, vol. 14 (1928), pp. 75–82, last paragraph.)

23. Dr. Lulu Hofmann and Professor Edward Kasner: *Homographic circles.*

This paper will appear in full in an early issue of this Bulletin.

24. Professor Edward Kasner: *The increment ratio of any two polygenic (or non-analytic) functions.*

In earlier papers (see Science, vol. 66 (1927), p. 581 and abstracts in this Bulletin), the author studied the derivative dw/dz, where $w=\phi(x, y)+i\,\psi(x, y)$, the components ϕ and ψ being arbitrary. In the present paper he studies dw_1/dw_2, where w_1 and w_2 are any such polygenic functions. To each point $z=x+iy$ corresponds a circle in the derivative plane, as in the simpler case, but the correspondence between the directions $m=dy/dx$ and the points of the circle is now a general homographic correspondence instead of being uniform (rate $-2:1$). The only cases in which dw_1/dw_2 is uniform are when w_2 is an analytic function of $x+iy$ or of $x-iy$, in the latter case the rate being $(2:1)$. If all the derivative circles reduce to points, w_1 must be an analytic function of w_2. To all the points of the z plane corresponds a congruence of homographic clocks. If a congruence is given arbitrarily, necessary and sufficient conditions are obtained for its being identifiable with a derivative congruence.

25. Professor T. R. Hollcroft: *The lines of an algebraic surface.*

In this paper, the number of invariants necessary and sufficient for a surface to contain a line or a given system of lines is found. From this is obtained the maximum number of independent lines or lines of a given system lying on an algebraic surface of given order. These limits hold when the surface is otherwise non-singular. When the surface has multiple points, lines, or curves, and these are accompanied by sets of lines on the surface, the lines of such sets do not account for additional invariants. The monoid is an example of this.

ARNOLD DRESDEN
Associate Secretary

A CORRECTION

On page 139 of the March-April issue (vol. 34), the last sentence in the abstract of paper No. 17 is incorrect and should have been omitted.

RESONANCE IN THE SOLAR SYSTEM*

BY E. W. BROWN

While the title of this address implies limited applications of resonance phenomena, it is necessary, in order to develop those applications, to consider the foundations on which the theory of resonance rests. The foundations are partly physical and partly mathematical since we have to consider not only the phenomena but the symbolic representation of them. At the very beginning of the discussion it is seen that questions concerning the degree of accuracy of physical measurements are involved in a fundamental manner. A wide range of topics has thus to be brought into the argument and some considerable time must be spent on quite elementary details if the ideas to be developed are to be made clear. The physicist will recognize applications in many directions, but I shall confine the latter mainly to those which arise in the motions of bodies within the solar system.

In order to make clear the questions with which we have to deal, it is necessary to understand what we mean by *periodicity*. Take the case of simple harmonic motion defined by

$$x = a \sin (bt - c).$$

In all physical problems, a, b, c are measured quantities deduced directly or indirectly from observation. If small errors occur in a, c only, the change in x is always small, however great t may be. But a small error in b induces finite changes in x when t is sufficiently great. As our power of measuring b is always limited, the time during which we can predict the value of x is also limited. Further, the conditions under which mechanical systems operate are such

* The fifth Josiah Willard Gibbs Lecture, read at Nashville, December 28, 1927, before a joint session of the American Mathematical Society and the American Association for the Advancement of Science.

that b itself will not remain permanently the same, unless it be a universal constant of nature, and we know of no such constant as far as the motions of finite masses are concerned. The same result must follow for any periodic motion. The mathematical definition of periodicity is thus not sufficient. In the applications we must regard it as an average or temporary phenomenon. This point of view has to be emphasized when, in celestial mechanics, we use the phrase "mean angular velocity" which implicitly assumes that such a mean exists and is determinate. Where we are dealing with phenomena over a limited time interval, the matter is not usually so important, but when our interval becomes very great as in questions of cosmogony it must be taken into account.

The word *resonance* is usually used to describe the phenomena which are observed when, in some mechanical system in motion, the periods of oscillation of two of its parts, not rigidly connected with one another, become the same. It would seem at first sight that the definition has no meaning, for the slightest change in our measures of the periods would abolish the resonance. This difficulty partly disappears when we learn that, in general, it is not necessary to have the periods exactly equal in order to produce the phenomena; it is only necessary that their difference shall be less than some finite amount. In the language of mathematics, the phenomena of resonance belong, not to a point, but to a finite range.

When we remember the fundamental importance of resonance in all physical observations—indeed our capacity to see and hear and therefore to observe depends on its existence—it is somewhat strange how little concerning the actual phenomena is to be found in the textbooks and treatises which deal with physical problems. Partly this is due to the fact that damping due to friction occurs in our mechanical systems and damping prevents the appearance of the essential characteristics of the motion. Partly also it is due to the general use of linear systems of differential

equations which give results differing fundamentally from those of non-linear systems. In the resonance problems of celestial mechanics we have no damping, the equations are not linear in general even in a first approximation, and every "constant" present is a measured constant and subject to both error and change. Since a considerable degree of delicate adjustment is required to produce resonance, it is evident that these factors must be taken into consideration when discussing the mathematical representation of the phenomena.

Let us take the linear system

$$\ddot{x} + n^2x = a \sin (pt + \alpha).$$

This represents a motion which is the sum of two harmonic oscillations of periods $2\pi/n$, $2\pi/p$, as long as $n \neq p$. Since the amplitude of the second oscillation is $a \div (n^2 - p^2)$, this amplitude increases indefinitely as $n-p$ approaches zero. When $n-p=0$ exactly, the solution changes its character and can be represented by two harmonic terms one of which has an amplitude of the form bt where b is finite. But from the physical point of view there is no discontinuity in passing from $n-p$ very small to $n-p=0$: the oscillations increase in amplitude as $n-p$ approaches zero.

For a linear system with several variables

$$\ddot{x}_i + \sum_j a_{ij}x_j = 0, \qquad (i.j = 1,2, \cdots, n),$$

in which the constants a_{ij} are such that each coordinate can be represented as the sum of n harmonic terms with n different periods, it can be shown that a similar increase in the amplitudes occurs, when two of the periods become nearly equal. An exception to this result arises when we can, by a linear change of the variables, separate the system into two independant sets in one of which $i, j = 1, \cdots, p$ and in the other, $i, j = p+1, \cdots, n$.

These results, however, are of little value when the systems are not linear. Both the physical characteristics

of the ensuing motions and their mathematical representation are changed. We cannot lay down any general type of equation, as in the linear case, and apply it to the particular cases which may be presented. The better plan therefore seems to be to take a special type and discuss it in some detail. But before doing so, I must try and give some indications of the difficulties which planetary and satellite motions in the solar system present, and of the methods which at present are in use to attack them.

The problem of the motion of three or more bodies under the Newtonian laws of motion and gravitation is so complicated that I cannot hope even to give an outline which shall be intelligible in the course of an hour's talk. And it would take me far longer to lead to the particular phenomena which form the subject matter of this lecture by any logical process even if the mere steps were simply outlined and explained. I shall therefore content myself with presenting the problems involved in the form of mathematical or physical analogies which will have the additional advantage of corresponding to simple physical mechanisms with which everyone is familiar.

I have two main ideas in view in discussing the question of resonance. One, the more immediate, is an attempt to deal with a set of observed phenomena which have not as yet received a fully adequate explanation, or perhaps, to be a little more specific, to account for the phenomena on the law of gravitation only without invoking special hypotheses as to the original formation of the solar system. This first question is the apparent avoidance of resonance in certain cases of planetary motions and the apparent permanence of resonance in others. The course of the discussion is seen to lead inevitably to the two parts of the second question—the configuration of the system in the distant past and distant future. Can we learn anything about the first part, usually termed the origin of the solar system, from its present configuration? And to what extent can we predict the future? Has the solar system the property which is usually but rather loosely, termed "stability"?

"Stability" in the usual sense is a mathematical rather than a physical property of an actual mechanical system. It is not difficult to deal with when the system is at rest, or when we confine its motion to certain very simple configurations. When we come, however, to deal with the problem of three or more bodies, it can have almost any meaning we like to attach to it. One common meaning in celestial mechanics is the limitation of one of the elements, say the velocity or mean distance, to certain limits. If we had any exact knowledge of what those limits were, there would be some satisfaction in using the term.

Let me give an illustration. One of the theorems often quoted is that which refers to the "stability" of the eccentricities. It is said that it can be shown that the eccentricities of the planets always lie between certain limits. What does this assume? The variables which define the motion are changed from the ordinary coordinates to new variables which are those defining the position, shape, and size of what is called the osculating ellipse. If there were only two bodies involved, this ellipse would have fixed position, shape, and size. When other bodies attract, these elements vary. Suppose we confine the variations to the first powers of the changes from constancy, as is done in the mathematical discussions on stability. Then the total change is the sum of all the individual changes. The theorem in question simply refers to the individual changes as calculated in this manner; it is the usual "mathematical stability." Each of them is periodic and therefore limited in magnitude. The proof, however, breaks down if we proceed to higher powers of the changes when there is approximate or exact resonance between two or more of the periods present. The limits between which the eccentricities can vary may become so wide as completely to alter the general configuration of the system; from the point of view of the physicist the motion is ultimately unstable.

In the ordinary discussions on stability we ask what is the result of slightly changing the position and velocity at any

instant. In resonance problems we have to ask what happens when we introduce small periodic disturbing forces; at least it is in this way that the problems of celestial mechanics often present themselves. The physicist accustomed to the phenomenon in most of his work is able to say at once that the oscillations of the system increase in amplitude, and this indeed may be regarded as the first general effect of resonance. What happens later depends on the construction of the system. In terrestrial mechanisms we are accustomed to observe the early increase of amplitude and the apparent change of period which accompanies it. Frictional or damping forces are, however, always present, and since these affect resonance phenomena in a fundamental manner it is not easy to use familiar illustrations in order to visualize what will happen in the absence of friction. In atomic systems, provided the periodic disturbing force is not too small, the description usually is confined to a statement that the type of motion changes. Atomic systems under their modern descriptions have certain resemblances to the problems of celestial mechanics. But there is this difference. In the atomic problems we observe the aggregate or average effect of millions of revolutions of the electrons about the nucleus but have no method of observing the positions in the orbits at any one time. In the solar system, we can observe the latter, but so far have not been able to calculate the result of millions of revolutions.

Let us now proceed to the physical analogy which will serve to illustrate some of the phenomena of resonance in the solar system. It depends on the fact that the mathematical expressions are somewhat similar when reduced to their simplest forms. As with most analogies, however, the correspondence is not complete and must not be pushed too far. The analog to be used is that of the motion of the common pendulum. But the pendulum I have in mind differs from that attached to a clock in that it can make complete revolutions. The mechanism is perhaps better thought of as a bicycle wheel mounted on a stand with

ball bearings so as to be as nearly frictionless as the imagination can make it; the actual description supposes that there is an entire absence of friction. To one point of the rim an iron weight is clamped. The wheel can be set in motion in various ways and can be acted on by different types of forces; some of them can be supposed to be impressed by a moving magnetic field which can act on the iron weight.

When no external force acts the motion of such a pendulum is well known. If it be given a small tap it oscillates in a period which depends on the amplitude; there is no such thing physically as an infinitesimal tap which gives a period independent of the amplitude. The greater the amplitude, the longer the period. The limit of the latter is infinite when the pendulum just reaches the highest point. The motion with a larger tap changes its character; the pendulum executes complete revolutions in either sense, with a variation of velocity between the highest and lowest points.

Mathematically, the motion is expressed by the equation

$$\ddot{x} + k^2 \sin x = 0,$$

where x is the angle between the position of the pendulum at any time and its equilibrium position at the lowest point. As long as the pendulum is oscillating, its motion is expressed by

$$x = \text{periodic function of } t, \text{ period } 2\pi/n.$$

When it is making complete revolutions, the motion is given by

$$x = \pm nt + A + \text{periodic function of } t, \text{ period } 2\pi/n.$$

In each case the period is an arbitrary constant of the solution depending on the initial impulse. It approximates to $2\pi/k$ only when the impulse is very small. Separating the two solutions is one in which x is a discontinuous (indeterminate) function of t at the highest point.

When the motion is near this critical case, a very small change in the initial conditions will not only completely alter the physical character of the motion but it will do so in a finite time. That is to say, the nearer the pendulum to the

critical case, the shorter is the interval during which we can predict its position with a given degree of accuracy. The comparison here is with the non-critical case where the accuracy of prediction is more nearly proportional to the accuracy with which we can measure the constants.

Suppose that while the pendulum is executing small oscillations a periodic force with a period nearly equal to $2\pi/k$ acts. We know what happens. The oscillations begin to increase and, after the amplitude reaches a maximum, begin to decrease again, the process being repeated as long as we are able to observe it. A force of this type might be expressed by

$$\ddot{x} + k^2 \sin x = a \sin (kt + \alpha).$$

The phenomena of resonance in this case may be described by saying that the amplitude increases until the average period is sufficiently different from $2\pi/k$ as to change the relative phases by 180°, when the amplitude begins to decrease again owing to the external force opposing the motion.

But if we ask whether this process is repeated indefinitely, the answer is not so simple. We may regard the system as having two periods, $2\pi/k$ and another period which approximately measures the time between maximum amplitudes. The motion cannot be exactly recurrent unless these two periods are in the ratio of two whole numbers. When this happens the equation must have a periodic solution. This would evidently be a special case and we are desirous of finding out what happens under *any* initial conditions. A way out of the difficulty is to suppose that we can always find a periodic solution.

Suppose that the two frequencies mentioned above were k, k'. The last hypothesis demands that we shall always be able to find two integers, i, j, such that $k/k' = i/j$. Suppose that we have found such a case and that we either slightly alter the initial conditions or, what amounts to the same thing, our measures from observation of k or k'. The ratio k/k' would then be i'/j', where i', j' are two new integers.

Now while the difference $i/j-i'/j'$ may be very small, the theory of convergents in continued fractions teaches that $i-i'$ and $j-j'$ will be widely different, and, in general the smaller the difference between the two fractions, the greater will be the differences between their numerators and their denominators. Since the whole periods of the motions are $2\pi i/k$, $2\pi i'/k$ in the two cases, the period is discontinuous; there is a finite change in it corresponding to an infinitesimal change in the initial conditions.

It may still happen, however, that the amplitude of x always remains less than some given quantity. Suppose we attempt to solve the equation by means of infinite series of harmonic terms. The arguments will be of the form $(ik \pm jk')t + \text{const.}$, where i, j are integers. The process of integration gives coefficients of the form $a_{ij} \div (ik \pm jk')$, where a_{ij} diminishes to zero as i or j approach infinity. In general, the values of a_{ij} change only by an infinitesimal amount when k or k' similarly changes, so that it would seem always possible so to choose the measures of k, k' that the series is finite or that it shall be infinite. This means that the motion is either physically indeterminate within a finite time, or that our assumption of a solution oscillating between finite limits must be abandoned.

The problem is similar to that of the integral

$$\int \sum a_{ij} \sin (ikt \pm jk't + \alpha_{ij}) dt, \qquad (i,j = 1,2, \cdots, \infty),$$

where $\sum |a_{ij}|$ is convergent. The result is infinite or indeterminate as long as k, k' are measured quantities. The only possible conclusion appears to be that, except for special cases, at some time in its motion the pendulum will begin to make complete revolutions. Incidentally, it is interesting to note that the small amount of friction present in the motion of the most carefully constructed pendulum is necessary in order that it may furnish an accurate measure of time.

Let us next consider what happens when complete revolutions are under way. The difficulties caused by the assumption of a doubly infinite series still appear to be present, that is to say, we cannot assume that the motion is expressed by $x = k_1 t +$ an oscillating function of t. In general, however, the average speed of the complete revolutions will not continuously increase but will reach a maximum and then diminish again. When they have reached the point where oscillations can take place, there are two possibilities, namely, oscillations, or complete revolutions in the opposite sense. But since a limitation of the phase of the motion is necessary in order that oscillations may be set up, the reversal of the complete revolutions will be the usual phenomenon and only after many reversals will oscillation be again set up. These, after continuing for a time, will again be changed to complete revolutions, the maximum speed of which is apparently not limited, but will itself have oscillations the extent of which will be greater at certain intervals, the longer the motion continues.

The times occupied by these various phases of the motion will depend on the values of the constants present in the motion. We can get some guide to these from the general principle that the body will remain longest in that phase of motion which has the greatest degree of relative stability, that is, where small changes in the constants will produce the least effect in a given interval of time. These appear in general to be either the configurations where the oscillations are at their minimum amplitude or where the revolutions have their maxima of average speed, or at both. In certain cases of celestial mechanics in which the mechanism is too complicated for the use of the pendulum as an approximate analogue, small oscillations are impossible, and for these the phases corresponding to the maximum speeds of revolution will be the configuration in which most of the time will be spent.

The general description of the motion of a pendulum under

a periodic force $a \sin (pt+\alpha)$ is not essentially different from that under a force

$$a \sin (x + pt + \alpha),$$

which resembles more closely some of the cases in celestial mechanics, especially when p/k is small. This latter is the case of a periodic force which has nearly the same period as the pendulum for all amplitudes except for those which approach complete revolutions. It can be shown that unless a/pk is small, the latter state is in general set up in an interval comparable with $2\pi/p$. A relatively stable configuration is not reached until the average angular velocity of revolution considerably exceeds p. When a/pk is small, the time required to produce complete revolutions may be very long. The distinction between the cases a/pk large and small appears to have an important bearing on the asteroid problem.

It has been stated that, in general, the motions which have the greatest relative stability are those in which the average speed of revolution considerably exceeds p. The ordinary processes of approximation show that in this motion, the periodic change in the average angular velocity as the pendulum goes round varies as k^2. As the speed diminishes, this amplitude tends to vary as the first power of k. In the applications to celestial mechanics, the unit of time is the period of revolution of the body,* and in all such cases k^2 is a quantity small compared with unity. Thus k^2 is small compared with k. Since resonance corresponds to the case where the pendulum is oscillating, it is convenient to define a "range" attached to the resonance and to measure it by k. Thus resonance phenomena take place within the range k; outside this range ordinary oscillations or perturbations, as they are usually called, occur.

I have dealt hitherto with the cases where the motion is analogous to a pendulum which can make very small

* This must not be confused with the period of revolution of the pendulum.

oscillations when no disturbing force acts, and have mentioned the possibility of cases of motion in which this analogy breaks down to the extent that small oscillations are impossible, while oscillations of finite amplitude do appear to be possible. It is evident that in this event the change from oscillations to complete revolutions under disturbing forces of the form $a\sin(x+pt+\alpha)$ will take place more rapidly and that the time during which oscillations are under way will be relatively shorter. This result may be put in a form more useful for the applications by the statement that the time spent in oscillation compared with that spent in complete revolutions is very much shorter, and consequently over any long interval the chance of observing oscillations is smaller.

In the whole of this discussion it is assumed that there is a complete absence of frictional forces. The moment we introduce them, however small in amount, the physical results are changed fundamentally. It would seem that the most familiar resonance effect—the identity of the periods of rotation about its axis and revolution around the earth of the moon—could not persist without large oscillations (librations) in the absence of small viscous forces in the moon's mass. The first effect of a non-uniform change in the period of revolution is to increase the amplitude of these free oscillations or librations. But the lunar bodily viscous forces tend to damp them down if the change is very slow. It follows as a corollary that once this resonance has been set up, it may continue indefinitely. Bodily viscosity in the moon's mass therefore changes a configuration which is unstable in the presence of small periodic disturbing forces into one in which the stability is permanent. Thus the moon will always turn the same face to the earth. A similar effect keeps the Eulerian nutations of the earth's axis small.*

My main object, however, is the consideration of resonances amongst planetary and satellite motions where

* H. Jeffreys, *The Earth*, p. 248.

we assume a complete absence of frictional forces, each body being treated as a particle with complete rigidity. I shall first consider an asteroid of very small mass moving in a planetary orbit round the sun, and acted on by planets of large masses, Jupiter and Saturn for example, the latter however, having masses small compared with that of the sun.

If the asteroid were not acted on by other planets it would describe an ellipse about the sun with constant period $2\pi/n$ and the angle described after any time would be given by

$$v = w + \text{periodic function of } w,\ w = nt + \text{const.},$$

the distance being a periodic function of w having a finite mean value. The "elements" of the ellipse are usually taken to be w, the mean distance, the eccentricity, and three further elements which define the position of the ellipse, and the position of the asteroid in the ellipse at some given moment. When Jupiter acts on the asteroid, one useful method is to find how these elements vary. Suppose Jupiter is moving in a fixed ellipse, the mass of the asteroid being too small to affect the motion of Jupiter sensibly. Then the motion of Jupiter is expressible as a function of $w' = n't + \epsilon'$, the other elements being constant.

The only way we have to find the motion of the asteroid under these circumstances is to proceed by some method of continued approximation. When we do this, the first approximation to w is given by an equation of the form

$$\ddot{w} = mn^2 \sum i a_{ij} \sin(iw + jw' + \beta_{ij}),$$

where $i, j = 0, \pm 1, \pm 2, \cdots$, giving an infinite series. The magnitudes of the coefficients mainly depend on a parameter e which is of the order of the eccentricities and inclination. In general, a_{ij} is divisible by $e^{|i+j|}$. The factor m is the ratio of the mass of Jupiter to that of the sun and is about .001.

Having decided what accuracy we need, we cut off the series at some definite values of i, j. We then put $w = nt + \epsilon$,

$w' = n't + \epsilon'$ in the right hand member and integrate, obtaining

$$w = nt + \epsilon - mn^2 \sum \frac{ia_{ij}}{(in + jn')^2} \sin (iw + jw' + \beta_{ij}).$$

This method gives a good approximation so long as $in + jn'$ is not too small.

Suppose, however, that $in + jn'$ in one term is so small that the corresponding coefficient is very large and that even the oscillation of the mean angular velocity, w, is large. Then evidently the approximation is useless.

To avoid the difficulty put

$$iw + jw' + \beta_{ij} = x + \pi \text{ or } x$$

according as a_{ij} is positive or negative, and omit all the other terms as being small compared with this. The equation for x then takes the form

$$\ddot{x} + k^2 \sin x = 0.$$

Since the integral of this equation is

$$\dot{x}^2 = b + 2k^2 \cos x,$$

the value of the angular velocity is limited whatever be the initial conditions. So far the presence of this resonance term produces no infinity. But the character of the motion is changed similarly to that of the pendulum when it changes from making oscillations to complete revolutions.

Suppose now that Saturn is present. It is large enough to affect the motion of Jupiter sensibly and introduces terms of the form $a \sin(x + pt + p')$ where $2\pi/p$ is a period which may be very long compared with the period of revolution of the asteroid. If this is simply an addition to w' its effect is not usually difficult to deal with for reasons into which I shall not enter. But there are certain terms which produce large periodic changes in the eccentricity of Jupiter—the so-called secular terms— and these terms appear to compel us to deal with equations of the form

$$\ddot{x} + k^2 \sin x = a \sin (x + pt + \alpha).$$

As long as a/pk is small, we can continue our approximations, and it may be that the higher approximations will avoid the phenomena which have been explained earlier, although this is unlikely. In any case, however, the oscillation, if started with x small, may continue for long periods of time before x can reach the values $\pm\pi$. But if a/pk is large, the oscillations must soon turn into complete revolutions and continue until the asteroid is well outside the resonance range. Once there, as explained earlier, it will at least remain for a long time even if it can get back at all.

From the point of view taken here, the most important group of asteroids is that known by the name "Trojan," from the circumstance that all its six known members have received names famous in the Iliad of Homer. The mean period of each of these round the sun is at present the same as that of Jupiter, that is, the resonance is due to the equality of n, n', and consequently the resonance terms are those for which $i=-j$. The attendant circumstances, namely, that they are at the same mean distance from the sun and that they oscillate about the third vertices of the two equilateral triangles which have Jupiter and the sun as the other two vertices, are incidental to this discussion, except in so far as they render the discussion possible. Laplace, a century ago, showed that an ideal exact solution of the problem of three bodies was a configuration in which all of them remained at the corners of an equilateral triangle while describing ellipses of the same size and shape. It has further been shown that under infinitesimal displacements from this ideal configuration these deviations will remain infinitesimal, under conditions as to the magnitudes of the masses which are here satisfied.

Now the set of deviations in which we are interested corresponds to the case of the pendulum performing infinitesimal oscillations. But the observed deviations are far from being of this character; in the case of one asteroid they correspond to an amplitude of the order of 20° from the vertical position.

If we continue the approximations on the supposition that Jupiter moves in a fixed elliptic orbit, there is nothing to indicate, within the extent to which these approximations can be carried with a reasonable amount of labor, that the configuration is otherwise than stable for very long intervals. When, however, we introduce the action of Saturn, both directly on the asteroid and indirectly through its effect on Jupiter, we introduce large perturbations, chiefly because these two planets and therefore Saturn and an asteroid, have their periods nearly in the ratio 2:5. It can be shown that the principal part of this action—that arising in the mean angles w, w'—is transferred directly to the asteroid. But there are other portions constituting slow changes in the eccentricity of Jupiter which introduce terms corresponding to the disturbing force $a \sin(x+pt+\alpha)$ in the motion of the pendulum. The number a/kp for these terms is small as long as the eccentricity of the asteroid is not too great. In the case of one asteroid, Achilles, I was able nevertheless to show* that a very slight change in the conditions would produce large oscillations (a secondary resonance) in its eccentricity—so large indeed that they might bring the asteroid to a close approach to Jupiter. Changes of this character are always furnished by the attractions of other planets, granted sufficient time, so that the present apparent stability of the orbit may be lost in the course of time.

The next great group is that of the 2:1 ratio. The first approximation for this group with Jupiter alone acting is such that small oscillations are impossible but oscillations with finite amplitude probably can exist. When we include the action of Saturn, the analogue of the case in which a/kp is large comes into play. We therefore expect that the time spent in oscillations is small compared with that spent in making complete revolutions. The "range" of the resonance is approximately $(0.0007e)^{1/2}$ where e is the average eccentricity.† If e be 0.1, this gives us a range of period for

* Transactions, Yale Observatory, vol. 3, p. 114.

† Monthly Notices, Royal Astronomical Society, vol. 72, p. 619.

resonance conditions of the order of fifty days about the half period of Jupiter. Within this range very few asteroids should be found at any one time. There is another feature about this resonance point which can be deduced from the equation, namely, that oscillations can only be set up from complete revolutions in one sense; this of course statistically cuts down the probable number by one half.

In all resonance cases except that of the 1:1 ratio the range of the eccentricities is large. In the case of the 2:1 group, in whatever manner the asteroid be started, the eccentricity at some time must be at least 0.2.* Near the edge of the range is a resonance point 5:1 with Saturn, and though the range of this new resonance point is much smaller, it will add to the improbability of finding an asteroid within the 2:1 range at any time. These results agree well with the statistical data. A considerable number of asteroids are known which can be well interpreted as having been nearly uniformly distributed near this resonance point, but there is a complete (with one or two doubtful exceptions) absence of asteroids within the range. Further, there appear to be more than the average number just outside the range.

The equations for all the resonance points $(1+i):i$ are quite similar to those of 2:1 with a slow decrease in range as i increases. There are, however, few asteroids near these ranges, so that statistical results are not properly applicable. But with higher values of i, two facts must be noted. In the first place, the resonance points become more crowded together as i increases and they approach the 1:1 resonance as a limit. In the second place, the mean distances approach that of Jupiter with increase of i and a circumstance of which little account has been yet taken, namely, a very close approach to Jupiter becomes more probable. Such a close approach fundamentally changes the orbit and more particularly the eccentricity, so that asteroids which have experienced this phenomenon must be regarded as belonging

* Loc. cit., Monthly Notices, p. 629.

to a somewhat different statistical group. The above considerations, however, indicate that the chance of an asteroid remaining near these higher resonance points diminishes rapidly with the increase of i, or in other words, the action of Jupiter and Saturn will clear them away much earlier than those near the 2:1 ratio. Finally, while the resonance points $i:j$ for small values of i, j are those most likely to be effective in showing resonance phenomena, the values of n'/n just outside these resonance ranges (except the 1:1 case) are the most stable since, when the ranges are taken into account, these positions are the least crowded of all by resonance phenomena.

Besides the asteroids, there is a family of comets which have also periods nearly half that of Jupiter. On account of their large eccentricities the methods used for the asteroids are not available. But it would seem that the same general results will hold. During the writing of this paper, the conclusions reached were illustrated in a publication by Professor A. O. Leuschner* on the Pons-Winnecke Comet. This comet in 1819 had a mean daily angular velocity round the sun of 632″ which increased to 639″ in 1858 and then diminished to 604″ in 1915. Between that date and 1927 it crossed the 2:1 resonance point of 599″ and is now 591″. This extraordinarily rapid change is partly "incidental" and due, as Leuschner points out, to the fact that the aphelion distance and phase are such that, during the last century, the comet has approached Jupiter very closely every other revolution. Theory suggests that these close approaches cannot persist through very many more revolutions. The point, however, namely that a large perturbation can carry a body rapidly through the resonance region, seems to be exemplified in this case.

Other resonance points within the group of asteroids are treated in the same general manner. In the case of the 3:1 group, small oscillations appear to be possible. The range of

* Publications of the Astronomical Society of the Pacific, October, 1927.

the resonance is smaller and the statistical material less plentiful with a smaller region of resonance; however, the disturbing effects have more power; in fact, in our pendulum analogy the terms of the right-hand member are of the same order of magnitude as those on the left, and, as stated above, the time which an asteroid can spend in this region is correspondingly less, giving a smaller chance of finding an asteroid there at any one time.

The rings of Saturn furnish valuable statistical material. Here the millions of "stones" or "rocks" which must constitute those rings revolve round Saturn and resonances are caused by the action of its larger satellites. There is little doubt that the well known divisions of the rings are caused by resonance action. But unless my main thesis is correct, namely that the time spent in oscillation within the resonance region is very short compared with the time spent outside, it is difficult to account for these divisions. Unfortunately we cannot at present observe the motion of individual members of the rings. We can, however, obtain a measure of the number at a given distance from Saturn by photometric observations and this number should be indicated also by the theory. The mathematical treatment, while very difficult and full of pitfalls, does not appear to be outside the range of present possibilities. It must, however, be carried out on practical lines, that is, with full consideration of the observational data.

I have up to now dealt with cases in which the integrals of the problem of three or more bodies are of little or no assistance on account of the small masses of the disturbed bodies in comparison with those of the bodies which produce the perturbations. Outside the integrals which define the motion of the center of mass of the system and which play no part in this discussion, there are only four known, or, if we confine the motion to one plane, only two, namely, the integrals of energy and angular momentum.

Since the portion contributed by a body to any one of these integrals has the mass of that body as a factor, it is

at once evident that very small changes in the momentum or energy of a large body can change fundamentally those of the small bodies, as far as these integrals are concerned. From the point of view taken here, an exact integral constitutes a limitation on the range of variation. This may also be interpreted by the statement that the existence or near existence of an integral lengthens the time during which a given configuration can persist. When the masses present in the integral are of the same order of magnitude we can make use of it. Thus the mutual action of Jupiter and Saturn can be treated with the use of these integrals because the actions of the other planets in the solar system affect the integrals only to a comparatively small degree in a limited time. Similarly, in testing the mutual actions of the satellites of Jupiter and those of Saturn, we can also make use of these integrals. Temporarily stable resonance conditions are therefore more probable, relatively to the number of bodies under observation, than in the asteroid problem. And in fact, such a relation between the motions of three of the inner satellites of Jupiter is known to exist at the present time. And there is another, of a more complicated character, between two of the satellites of Saturn.

The case of Jupiter and Saturn is an interesting one in this connection, because the ratio of their periods is nearly 2:5. Another is that of Neptune and Uranus whose periods are nearly 2:1. While they are well outside the resonance "ranges," it is not possible to assert that they never have been or never will be within these ranges. Neither case permits small oscillations within the ranges. It is indeed entirely possible that they illustrate again the fact mentioned above, namely, that the positions which have the highest temporary stability are those just outside the largest resonance ranges. This fact has been pointed out as an observational result by previous workers. My argument here has been an attempt to show that these are probably not configurations of permanent stability, but merely those in which

a much longer time will be spent than within the resonance regions.

A few words in reference to the application of the theory of resonance to the orbit of the moon may conclude this part of my subject. It has, owing to its proximity to the earth, received far more attention from the celestial mechanician than any other body, and its motion has been calculated to a high degree of accuracy. This calculated motion, involving at least ten arguments derived independently from observation, contains many hundreds of linear combinations of these arguments and thousands of other combinations have been examined. In no case has a combination giving a resonance with an observable coefficient been found. And yet, owing to tidal friction we know that its mean angular velocity is subject to a real secular change. On the law of chance alone we should not expect this result. If, however, the result obtained above is correct, namely, that a resonance condition with a large periodic perturbation moves the body away from the region rapidly, the chances of finding it there are much diminished. The changes in mean angular velocity due to tidal friction would therefore not be steady over periods measured by millions of revolutions but would contain intervals where small changes took place with comparative rapidity, the longer intervals of slow change being similar to that which characterizes the motion of the moon at the present time.

What general conclusions can we draw from these configurations? The first and the most obvious is that our knowledge of the past or future of the solar system must be a function of the interval within which it has been under observation, so far as the law of gravitation alone is concerned. The absence of a knowledge of any integrals of the problem of three or more bodies, beyond those known, is partly responsible for this result. If we try to represent the motion by oscillating functions we are met with the difficulty that these give an indeterminate result beyond a certain degree of accuracy. It is possibly true that there are no

secular changes in the strict sense in any of the linear coordinates, but it seems equally true that, owing to resonance, we cannot lay down any limit to the amplitudes of these oscillations and thus limit the changes of the present configuration. There appears to be no regular increase in these amplitudes; it becomes rather a question of probability than of calculation as to whether any particular configuration will be fundamentally changed within a very long interval of time. It seems necessary that the smaller bodies of the solar system should have orbits which are under continuous development and this necessarily carries with it a continuous development of the whole system.

One way out of the difficulty which has been sometimes suggested is the adoption of a "quantum" frequency which would permit of complete periodicity. This frequency, however, would have to correspond to a period much longer than the probable past history of the solar system. To be logical it should extend to the whole material universe and should therefore be independent of the amount of matter present, that is, it should be a universal constant of nature—a property of space in the sense of Einstein. The assumption has philosophic interest only since it appears to have no practical bearing on the actual problem on account of the time factor involved and of other reactions of matter which are continually altering the conditions of motion.

This last remark is equally applicable to assumptions concerning the existence of periodic solutions corresponding to an arbitrary set of initial conditions; such solutions, if they exist, will have periods too long for us to be able to say that the non-gravitational forces have not had a sensible effect in the period. It does not seem possible to avoid the conclusion that small changes in the motion at any time will ultimately produce finite changes in the configuration, the amount of the change being a function of the interval, as well as of the other conditions. This amounts to the statement that the solar system, and indeed any gravitational system consisting of several bodies, is always in a state of

development for gravitational reasons alone. The question is therefore shifted to that of the speed of this development, and I have attempted to show that in this speed the phenomena of resonance play a considerable part. Accumulated evidence goes to show that the history of the solar system from its formation into discrete bodies is much greater than 10^9 years. During such an interval it appears to be doubtful whether we can deduce much as to the initial configuration from the present configuration, except what the integrals of energy and angular momentum may furnish.

Certain of the present statistical distributions appear to be the results of chance in the same sense that the result of flipping a coin is said to be due to chance. In other words, a limited amount of information is to be obtained from considerations of probability. Thus the few known asteroids of the Trojan group are still there because the initial conditions of a large number presumed to have been there originally are such that the chance of still finding some there is not too small. We find few or none in the 2:1 and 3:1 ranges and perhaps in other resonance ranges because the chance of finding one under oscillation conditions is very small. The dark rings of Saturn are partly denuded of members of the system for the same reason. I am inclined to believe that the remarkable "families" of asteroid orbits deduced by K. Hirayama* will ultimately be found to be the results of perturbation effects due to resonance points with large ranges, rather than to some original configuration in which the bodies were all portions of some single body; in fact that "families" are due to the relatively stable regions just outside the resonance regions with large ranges. They are observable because the laws of chance indicate that at any one moment more asteroids will be found there than elsewhere.

It has already been pointed out that under resonance conditions, and especially those of the 2:1 ratio, the eccentricities will vary greatly. If a large number of bodies

* Japanese Journal of Astronomy, Vol. 1, No. 3.

were started near such a region, the chance of one of them getting close to Jupiter on account of large eccentricity is greatly increased. Of these a certain proportion will have their eccentricities so greatly altered as to move afterwards in cometary orbits. The periods will, in general be much less changed. On this basis, the numerous short-period comets with periods nearly half that of Jupiter would be regarded as a succeeding stage of the evolution of asteroids rather than a previous stage as suggested by Leuschner in his paper on the Pons-Winnecke Comet already referred to. The close approach to Jupiter would turn loose surface material on the asteroid into satellite orbits about the asteroid and thus perhaps furnish the material for the "tail", further loosening of the material resulting from the temperature changes taking place at each near approach to the sun. Some evidence in favor of this hypothesis is furnished by the apparent dissolutions of two or more comets which have taken place within historic times. It is difficult to account for this last fact except on the hypothesis that the supply of comets is being renewed. The same mode of development should apply to the rings of Saturn. There, however, an eccentric "stone" is liable to collide with other members of the system before it gets very close to one of the satellites, and thus a certain average circularity would be preserved.

The conclusion that the calculus of probabilities is more likely to lead to further information than the logical processes of exact analysis is perhaps a confession of defeat. For probability itself is here nothing but the chance of our being correct or incorrect in any statement, so long as we suppose that a determinate set of initial conditions will always produce a determinate result. When, however, we remember that our conditions are never exactly determinate in the mathematical sense on account of the inevitable errors of measurement, and that a very small change in the measurements will ultimately produce a finite change in the calculated configuration, this procedure is perhaps the only way in which we can deduce the needed information.

I am very fully aware of the unsatisfactory nature of an address supposed to be of a mathematical and therefore logical character which, at least partly, contains speculations as to mathematical or physical results which would seem to be capable of being logically deduced from the data. The reason for this procedure is the present condition of investigation in celestial mechanics. While a subject is rapidly advancing under the combined attacks of theory and experiment, speculations published without a sufficient background of observation are often harmful to progress because they diminish the interest of those who, equally capable of making the same speculations, take time to test them by experiment or calculation. But where a subject seems to have reached a condition where little progress is being made, and this I think is true as far as the gravitational development of the solar system is concerned, some speculation may be useful, if it indicates new avenues of approach or a more thorough exploration of old avenues. The speculations indulged in here have, however, a basis of physical and mathematical analogies and of experience gained from special examples. The processes followed have indicated how certain observed phenomena can be explained and have indicated also certain limits to the amount of information which can be obtained from the methods at present in use. The arguments lack precision; one question raised is whether it is possible to obtain precision. Our capacity to calculate and measure is limited, and it is this capacity which seems to be a part of the general question. In any case, it cannot be laid aside when we desire to obtain information as to the remote past or remote future of the solar system.

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SOME CANONICAL FORMS AND ASSOCIATED CANONICAL EXPANSIONS IN PROJECTIVE DIFFERENTIAL GEOMETRY*

BY E. B. STOUFFER

1. *Introduction.* A simplification of the methods of approach to any branch of mathematics is always very desirable. This is particularly true in a geometry, where extensive analytical machinery must be set up before geometric results can be obtained. This paper is a contribution to the simplification of Wilczynski's methods of attack upon plane and space curves in projective differential geometry.†

A canonical form for the fundamental differential equation associated with the curve is determined. It leads at once to a complete and independent system of invariants and covariants in their canonical form. The determination of the corresponding system in the general form involves only simple substitutions. An associated canonical expansion for the equation or equations of the curve is obtained by very direct methods and the geometrical significance of the corresponding triangle or tetrahedron of reference becomes easily evident. Because of the method of their derivation, the fundamental invariants and covariants obtained are those which have an immediately evident geometrical significance.

The methods here employed may be applied to surfaces, both curved and ruled in ordinary space and may also be extended to geometry in hyperspace. The resulting simplifications are in some cases quite remarkable. These results will be presented in later papers.

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† See Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces*, Chapters 2, 3 and 13.

In a series of papers* published in recent years, Sannia has made a study of plane and space curves by methods related somewhat to those used in this paper. He employs the absolute calculus but the results obtained below show that its use is unnecessary in order to gain real simplicity. Moreover, the methods of Sannia by no means extend themselves so naturally and so simply to surfaces in ordinary space and to hyperspace.

2. *Plane Curves.* It is well known that all the projective differential properties of plane curves may be studied by means of a single differential equation of the form

$$y''' + 3p_1y'' + 3p_2y' + p_3y = 0, \tag{A}$$

where differentiation is with respect to the independent variable x and where p_i are functions of x. If the homogeneous coordinates of a curve C are three functions $y_i (i=1, 2, 3)$ of x then the coefficients p_i may be so determined that y_i form a fundamental set of solutions of (A), provided merely that C is not a straight line. Moreover, any other set of fundamental solutions $z_i (i=1, 2, 3)$ of (A) may be expressed in the form

$$z_i = \sum_{j=1}^{3} c_{ij}y_j, \quad (i = 1,2,3), \quad |c_{ij}| \neq 0.$$

Consequently all the curves determined by (A) in this way are projective transformations of C.

However, the parametric representation of C is not unique. A transformation of the independent variable of the form

$$\bar{x} = \phi(x), \tag{1}$$

and of the dependent variable of the form

$$y = \lambda\bar{y}, \tag{2}$$

where ϕ and λ are arbitrary functions of x, will not change the curve. Moreover, the form of (A) will not be changed

* Rendiconti dei Lincei, (5), vol. 31, 1° sem., pp. 450–454 and pp. 503–506; (5), vol. 31, 2° sem., pp. 17–19 and pp. 432–434; Annali di Matematica Pura ed Applicata, (4), vol. 1, pp. 1–18; (4), vol. 3, pp. 1–25.

by the above transformations but it will be converted into an equation with new coefficients. A function of the new coefficients and the new dependent variable and their derivatives which is equal, except for a factor, to the same function of the original coefficients and the original dependent variable and their derivatives is said to be a relative *covariant.* If the function does not contain the dependent variable or its derivatives, it is said to be an *invariant.*

The effect of the transformations is important. From (1) we have

$$(3)\qquad \begin{cases} y' = \dfrac{dy}{d\bar{x}}\phi', \; y'' = \dfrac{dy^2}{d\bar{x}^2}(\phi')^2 + \dfrac{dy}{d\bar{x}}\phi'', \\ y''' = \dfrac{d^3y}{d\bar{x}^3}(\phi')^3 + 3\dfrac{d^2y}{d\bar{x}^2}\phi'\phi'' + \dfrac{dy}{d\bar{x}}\phi''', \end{cases}$$

whence, by substitution into (A), we obtain a new equation

$$(4)\qquad \frac{d^3y}{d\bar{x}^3} + 3\bar{p}_1\frac{d^2y}{d\bar{x}^2} + 3\bar{p}_2\frac{dy}{d\bar{x}} + \bar{p}_3 y = 0,$$

where

$$(5)\qquad \begin{cases} \bar{p}_1 = \dfrac{1}{\phi'}\left(p_1 + \dfrac{\phi''}{\phi'}\right), \\ \bar{p}_2 = \dfrac{1}{(\phi')^2}\left(p_2 + p_1\dfrac{\phi''}{\phi'} + \dfrac{\phi'''}{3\phi'}\right), \; \bar{p}_3 = \dfrac{1}{(\phi')^3}p_3. \end{cases}$$

Likewise the transformation (2) gives

$$(6)\qquad \begin{cases} y' = \bar{\lambda}\bar{y}' + \bar{\lambda}'\bar{y}, \; y'' = \bar{\lambda}\bar{y}'' + 2\bar{\lambda}'\bar{y}' + \bar{\lambda}''\bar{y}, \\ y''' = \bar{\lambda}\bar{y}''' + 3\bar{\lambda}'\bar{y}'' + 3\bar{\lambda}''\bar{y}' + \bar{\lambda}'''\bar{y}, \end{cases}$$

whence by substitution into (4) we obtain a new equation in $\bar{y}$ whose coefficients $\overline{P}_i$ are expressed by the equations

$$(7)\qquad \begin{cases} \bar{\lambda}\overline{P}_1 = \bar{\lambda}\bar{p}_1 + \bar{\lambda}', \; \bar{\lambda}\overline{P}_2 = \bar{\lambda}\bar{p}_2 + 2\bar{\lambda}'\bar{p}_1 + \bar{\lambda}'', \\ \bar{\lambda}\overline{P}_3 = \bar{\lambda}\bar{p}_3 + 3\bar{\lambda}'\bar{p}_2 + 3\bar{\lambda}''\bar{p}_1 + \bar{\lambda}''', \end{cases}$$

where differentiation is with respect to $\bar{x}$.

All of the above is well known theory.* However, for the purposes of the present paper it is important to notice that equations (3), (5), (6), (7) show the following significant fact. *If by means of* (1) *and* (2) *equation* (A) *is transformed into a canonical form which is preserved if and only if*

$$\phi'' = 0, \quad \frac{d\bar{\lambda}}{d\bar{x}} = 0, \tag{8}$$

then the new variable $\bar{y}$ *and the new coefficients* $\overline{P}_i$ *and their derivatives are determined except for multiplication by certain factors.*

The resulting variable $\bar{y}$ and its derivatives will each be the canonical form of a relative covariant and must each determine a unique point in the plane. Likewise, the resulting coefficients $\overline{P}_i$ and their derivatives must each be the canonical form of a relative invariant. We shall see later that the determination of the expressions for these invariants and covariants in terms of the coefficients and variables of (A) will involve only simple substitutions by means of equations (3), (5), (6), (7).

The determination of the transformations which give a canonical form to our equations may be made in many ways and many different canonical forms may be obtained. It is evident that each such canonical form must lead directly to an equation for the curve in the form of an expansion of one non-homogeneous coordinate in terms of the other. Since such an expansion determines the geometrical properties of the curve, we naturally desire that the canonical form of (A) be such as to simplify the process of obtaining these properties from the associated expansion.

We now assume that our fundamental differential equation has been transformed into such a canonical form with coefficients $\overline{P}_i$ and variables† $\bar{x}$ and $\bar{y}$ and proceed to calculate

* See Wilczynski, loc. cit., Chapter III.

† In the remainder of this paper a bar above a letter will indicate that it represents a function of $\bar{x}$ and that differentiation of the function is with respect to $\bar{x}$.

the corresponding canonical expansion. It is an important fact that we can proceed just as if the conditions to be imposed were actually known.

Let $\bar{y}_i (i=1, 2, 3)$ be the coordinates of a regular point* y on C. We may assume without loss of generality that this point y is given by the value $\bar{x}=0$ of the parameter. Then by the general theory of differential equations the coordinates $\overline{Y}_i$ of any point $\overline{Y}$ on the curve in the neighborhood of $\bar{y}$ may be expressed as a power series in $\bar{x}$ of the form

$$\overline{Y} = \bar{y}(0) + \bar{y}'(0)\bar{x} + \bar{y}''(0)\frac{\bar{x}^2}{2} + \bar{y}'''(0)\frac{\bar{x}^3}{3!} + \cdots .$$

When we start to substitute for $\bar{y}'''(0), \bar{y}^{iv}(0), \cdots$ by means of the assumed canonical form of our differential equation, we see at once that a great simplification will take place if $\overline{P}_1=0$. The first of equations (7) shows that we can make $\overline{P}_1=0$ by choosing $\bar{\lambda}$ to satisfy the equation $\bar{\lambda}\bar{p}_1+\bar{\lambda}'=0$. Moreover, equations (5) and (7) show that the condition $\overline{P}_1=0$ will be maintained if (8) is satisfied. We shall therefore assume that $\overline{P}_1=0$ is one of the conditions imposed in order to give us our canonical form. Thus the canonical form is of the nature

$$\bar{y}''' + 3\overline{P}_2\bar{y}' + \overline{P}_3\bar{y} = 0, \tag{9}$$

where the coefficients are determined by the substitution of $\bar{\lambda}' = -\bar{\lambda}\bar{p}_1$ into (7) and have the form

$$\begin{cases} \overline{P}_2 = \bar{p}_2 - \bar{p}_1' - \bar{p}_1^2, \\ \overline{P}_3 = \bar{p}_3 - 3\bar{p}_1\bar{p}_2 + 2\bar{p}_1^3 - \bar{p}_1''. \end{cases} \tag{10}$$

The substitution from (5) into (10) gives

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* Throughout this paper a point with coordinates $\alpha_i (i=1, 2, 3)$ in the plane or with coordinates $\alpha_i (i=1, 2, 3, 4)$ in space will be denoted simply by α when no confusion can arise.

$$(11)\qquad \begin{cases} \overline{P}_2 = \dfrac{1}{(\phi')^2}\left(P_2 - \dfrac{2}{3}\psi' + \dfrac{1}{3}\psi^2\right), \\ \overline{P}_3 = \dfrac{1}{(\phi')^3}(P_3 - 3\psi P_2 + 3\psi\psi' - \psi^3 - \psi''), \end{cases}$$

where $\psi = \phi''/\phi'$ and where P_2 and P_3 are the same functions of p_i as are $\overline{P}_2$ and $\overline{P}_3$ of $\bar{p}_i$. The expressions for the derivatives of $\overline{P}_2$ and $\overline{P}_3$ are obtained from (11) by simple differentiation. The particular value of ψ which gives the canonical form is as yet undetermined.

Substitution from (9) into the above expansion now gives*

$$(12)\qquad \begin{cases} \overline{Y} = \bar{y} + \bar{y}'\bar{x} + \bar{y}''\dfrac{\bar{x}^2}{2} - (3\overline{P}_2\bar{y}' + \overline{P}_3\bar{y})\dfrac{\bar{x}^3}{3!} \\ \quad - [3\overline{P}_2\bar{y}'' + (3\overline{P}_2' + \overline{P}_3)\bar{y}' + \overline{P}_3'\bar{y}]\dfrac{\bar{x}^4}{4!} \\ \quad - [(6\overline{P}_2' + \overline{P}_3)\bar{y}'' + (2\overline{P}_3' + 3\overline{P}_2'' - 9\overline{P}_2^2)\bar{y}' \\ \quad + (\overline{P}_3'' - 3\overline{P}_2\overline{P}_3)\bar{y}]\dfrac{\bar{x}^5}{5!} \\ \quad - [(9\overline{P}_2'' + 3\overline{P}_3' - 9\overline{P}_2^2)\bar{y}'' \\ \quad + (3\overline{P}_3'' + 3\overline{P}_2'' - 36\overline{P}_2\overline{P}_2' - 6\overline{P}_2\overline{P}_3)\bar{y}' \\ \quad + (\overline{P}_3'' - 9\overline{P}_2'\overline{P}_3 - \overline{P}_3^2 - 3\overline{P}_2\overline{P}_3')\bar{y}]\dfrac{\bar{x}^6}{6!} + \cdots. \end{cases}$$

Since we are assuming that we have a canonical form, $\bar{y}_i$, $\bar{y}_i'$, $\bar{y}_i''$ are the homogeneous coordinates of three fixed points not on a straight line. Consequently, the coordinates of any point in the plane may be expressed in the form $x_1\bar{y}_i + x_2\bar{y}_i' + x_3\bar{y}_i''$. It follows that the coordinates of the point may be taken to be (x_1, x_2, x_3). The coordinates x_i of $\overline{Y}$ in this new coordinate system may be read directly from equation (12) and have the form of expansions in powers of $\bar{x}$.

* It is to be understood that in the coefficients of this expansion and those that follow $\bar{x} = 0$.

If we now pass to non-homogeneous coordinates we obtain the following results:

$$(13)\quad \begin{cases} \xi = \dfrac{x_2}{x_1} = \bar{x} - \dfrac{1}{2}\overline{P}_2\bar{x}^3 - \dfrac{1}{8}(\overline{P}_2' - \overline{P}_3)\bar{x}^4 \\ \qquad + (3\overline{P}_3' - 3\overline{P}_2'' + 9\overline{P}_2\,)\dfrac{\bar{x}^5}{5!} + \cdots, \\ \eta = \dfrac{x_3}{x_1} = \dfrac{\bar{x}^2}{2} - \dfrac{1}{8}\overline{P}_2\bar{x}^4 - (6\overline{P}_2' - 9\overline{P}_3)\dfrac{\bar{x}^5}{5!} \\ \qquad + (9\overline{P}_2{}^2 + 12\overline{P}_3' - 9\overline{P}_2'')\dfrac{\bar{x}^6}{6!} + \cdots. \end{cases}$$

The elimination of $\bar{x}$ from these equations gives us the desired expansion

$$(14)\quad \begin{cases} \eta = \dfrac{1}{2}\xi^2 + \dfrac{3}{8}\overline{P}_2\xi^4 + \dfrac{1}{40}(3\overline{P}_2' - 2\overline{P}_3)\xi^5 \\ \qquad + (9\overline{P}_2'' - 6\overline{P}_3' + 405\overline{P}_2{}^2)\dfrac{\xi^6}{6!} + \cdots. \end{cases}$$

We return now to the problem of the determination of our canonical form. It is easy to see from equations (11) that the term $3\overline{P}_2' - 2\overline{P}_3 = \bar{\theta}_3$ in the coefficient of ξ^5 in (14) is such that

$$\bar{\theta}_3 = \frac{1}{(\phi')^3}(3P_2' - 2P_3) = \frac{1}{(\phi')^3}\theta_3.$$

Consequently, assuming that θ_3 does not vanish identically,* we can make $\bar{\theta}_3 = 1$ by selecting $(\phi')^3 = \theta_3$. It is evident that the two conditions $\bar{\theta}_3 = 1$ and $\overline{P}_1 = 0$ are preserved only if $\phi'' = 0$, $\bar{\lambda}' = 0$. We then have for a canonical form of (A)

$$(15)\qquad \frac{d^3\bar{y}}{d\bar{x}^3} + 3\overline{P}_2\frac{d\bar{y}}{d\bar{x}} + \overline{P}_3\bar{y} = 0,$$

with $\bar{\theta}_3 = 1$, and for the associated canonical expansion

* It is easily seen that the identical vanishing of θ_3 is the condition that C be a conic. The projective differential properties of conics are of no interest.

$$(16) \qquad \eta = \frac{1}{2}\xi^2 + \frac{3}{8}\overline{P}_2\xi^4 + \frac{1}{40}\xi^5 + \frac{9}{16}\overline{P}_2\xi^6 + \cdots .$$

The coefficients $\overline{P}_2$ and $\overline{P}_3$ and their derivatives are all canonical forms of relative invariants. In order to obtain their general form it is only necessary to substitute the value of ψ obtained from $(\phi')^3=\theta_3$ into equations (1) and their derivatives. That the expressions thus obtained are relative invariants becomes evident if we take any two equations of form (A) which are equivalent under (1) and (2) and reduce each of them to our canonical form. The coefficients thus obtained can differ only by a factor and consequently the expressions for these coefficients in terms of the original coefficients will have the same form and can differ only by a factor.

That the invariants thus obtained together with θ_3 form a complete system is evident. In fact, since $3\overline{P}_2' - 2\overline{P}_3 = 1$ we actually need only the invariants arising from $\overline{P}_2$ and its derivatives together with θ_3 in order to have a complete system. That all the invariants in the latter complete system are independent is also evident.

Exactly the same type of argument shows that $\bar{y}$, $\bar{y}'$, $\bar{y}''$ are the canonical forms of relative covariants which may be expressed in terms of the original coefficients and variables by means of (3), (5), (6), (7) with the now known values of ϕ'' and $\bar{\lambda}'$ substituted. That the covariants thus obtained are independent is obvious. That they form with the complete system of invariants a complete system of covariants follows from the fact that the higher derivatives of $\bar{y}$ than the second can be replaced by means of equation (15).

We have thus obtained a complete and independent system of invariants and covariants from our canonical form by a process which involves nothing more difficult than direct substitutions.

The three fundamental covariants $\bar{y}$, $\bar{y}'$, $\bar{y}''$ determine precisely the three vertices of the tetrahedron of reference. The geometric determination of these vertices is not

difficult and gives results obtained by Wilczynski and Sannia. Equation (16) gives, by the method of undetermined coefficients, for the equation of the conic osculating C at $\bar{y}$

$$2x_3(x_1 - (3/2)\overline{P}_2x_3) - x_2^2 = 0, \tag{17}$$

and for the equation of the cubic osculating C at $\bar{y}$ and having a node at that point

$$10x_2x_3(x_1 - (3/2)\overline{P}_2x_3) - 5x_2^3 - 2x_3^3 = 0. \tag{18}$$

The tangent at (1, 0, 0), which is the point $\bar{y}$, to one branch of the cubic (18) is the line $x_2=0$ and the pole of this line with respect to the conic (17) is (0, 1, 0). Since (0, 1, 0) is the point $\bar{y}'$ and (0, 0, 1) is the point $\bar{y}''$, it is evident that the latter vertex is the point of intersection of the line $x_2=0$ with the tangent at $\bar{y}'$ to the curve generated by $\bar{y}'$.

One of the tangents from (0, 1, 0) to the conic (17) is $x_3=0$. The other tangent meets the cubic (18) in three distinct points, any one of which points may be used to determine the unit point of our system of coordinates.

Equation (15) shows at once the geometrical significance of the vanishing of the two fundamental invariants $\overline{P}_2$ and $\overline{P}_3$. If $\overline{P}_3=0$ the curve generated by the point $\bar{y}'$ is a plane curve. If $\overline{P}_2=0$ the tangent at $\bar{y}''$ to the curve generated by that point passes through $\bar{y}$, that is, the curve generated by $\bar{y}''$ is the envelope of the line $x_2=0$.

It is interesting to note at this point that essentially the canonical expansion used by Wilczynski is obtained if the vertex (0, 0, 1) is transformed to the second point of intersection of the line $x_2=0$ with the osculating conic.

All the machinery necessary for the study of the projective differential properties of plane curves has now been set up.

3. *Space Curves.* The projective differential properties of space curves can be studied by means of a differential equation of the fourth order of the form

$$y^{(iv)} + 4p_1y''' + 6p_2y'' + 4p_3y' + p_4y = 0. \tag{B}$$

The transformations of the variables which are permitted are exactly (1) and (2). In fact the methods employed and the equations derived in order to obtain the canonical expansion are obvious extensions of the methods and equations for plane curves. In place of equation (14) above we obtain in this case two equations

$$(19)\quad \begin{cases} \eta = \dfrac{1}{2}\xi^2 - \dfrac{1}{4}\overline{P}\xi^4 - 2(3\overline{P}_2' - 8\overline{P}_3)\dfrac{\xi^5}{5!} \\ \qquad - (6\overline{P}_2'' - 16\overline{P}_3' + 10\overline{P}_4 - 36\overline{P}_2^2)\dfrac{\xi^6}{6!} + \cdots, \\ \zeta = \dfrac{1}{6}\xi^3 - \dfrac{1}{20}\overline{P}_2\xi^5 - \dfrac{1}{180}(3\overline{P}_2' - 14\overline{P}_3)\xi^6 + \cdots. \end{cases}$$

In these equations $\overline{P}_2$, $\overline{P}_3$, $\overline{P}_4$ are the coefficients of the canonical form of equation (B) concerning which we know as yet only that $\overline{P}_1 = \bar{p}_2 - \bar{p}_1' - \bar{p}_1^2 = 0$. The condition which shall be imposed to produce $\bar{p}_i$ from p_i remains to be determined. In the above equations

$$\xi = \frac{x_2}{x_1},\ \eta = \frac{x_3}{x_1},\ \zeta = \frac{x_4}{x_1},$$

where the homogeneous coordinates (x_1, x_2, x_3, x_4) of a point in space follow from the coordinates of the point expressed in the form

$$x_1\bar{y}_i + x_2\bar{y}_i' + x_3\bar{y}_i'' + x_4\bar{y}_i'''.$$

Equations (19) do not indicate directly the conditions to be imposed in order to obtain a canonical form. In order to link our canonical form with the geometry we shall first calculate the osculating cubic, the simplest osculating curve other than the tangent line. Since a space cubic is determined by six points, the canonical expansion for the cubic osculating C at $\bar{y}$ must agree with (19) up to and including the fifth powers of ξ.

It is easily seen that the parametric representation of the osculating cubic can be put into the form

$$(20)\qquad \begin{cases} x_1 = 1 + a_1 t + b_1 t^2 + c_1 t^3, \\ x_2 = t + b_2 t^2 + c_2 t^3, \\ x_3 = b_3 t^2, \\ x_4 = c_4 t^3. \end{cases}$$

From these equations the expansions for $\eta = x_3/x_1$ and $\zeta = x_4/x_1$ in terms of $\xi = x_2/x_1$ are easily obtained. If the coefficients of the necessary powers of ξ are equated to the corresponding powers in (19), we obtain for the parametric equations of the osculating cubic

$$(21)\qquad \begin{cases} x_1 = 1 + \dfrac{9}{10}\overline{P}_2 t^2 - \dfrac{1}{30}(3\overline{P}_2' - 8\overline{P}_3)t^3, \\ x_2 = t + \dfrac{7}{10}\overline{P}_2 t^3, \qquad x_3 = \dfrac{1}{2}t^2, \qquad x_4 = \dfrac{1}{6}t^3, \end{cases}$$

and for the corresponding expansions for the cubic,

$$(22)\qquad \begin{aligned} \eta &= \frac{1}{2}\xi^2 - \frac{1}{4}\overline{P}_2\xi^4 - 2(3\overline{P}_2' - 8\overline{P}_3)\frac{\xi^5}{5!} + \frac{1}{200}\overline{P}_2{}^2\xi^6 + \cdots, \\ \zeta &= \frac{1}{6}\xi^3 - \frac{1}{20}\overline{P}_2\xi^5 - \frac{1}{90}(3\overline{P}_2' - 8\overline{P}_3)\xi^6 + \cdots. \end{aligned}$$

It now becomes evident that the projections of C and the osculating cubic from (0, 0, 0, 1), or $\bar{y}'''$, upon the osculating plane $x_4 = 0$ will have contact of the sixth order if and only if the coefficients of ξ^6 in the first equation of (19) and the first equation of (22) are equal, i.e., if and only if

$$\overline{M} = 15\overline{P}_2'' - 40\overline{P}_3' + 25\overline{P}_4 - 81\overline{P}_2{}^2 = 0.$$

But the equations for the space curve corresponding to (11) for the plane curve show that

$$\overline{M} = \frac{1}{(\phi')^4}(M + 15\psi\theta_3),$$

where*

* The expressions for $\overline{P}_i$ in terms of $\bar{p}_i$ are obtained by the substitution of $\bar{\lambda}' = -\bar{\lambda}\bar{p}_1$ into the equations for the space curve corresponding to (7)

$$M = 15P_2'' - 40P_3' + 25P_4 - 81P_2^2, \quad \theta_3 = 3P_2' - 2P_3.$$

It follows at once that, if $\theta_3 \neq 0$, we can make $\overline{M} = 0$. The conditions $\overline{M} = 0$ and $\overline{P}_1 = 0$ determine our canonical form.

The coefficients $\overline{P}_2$, $\overline{P}_3$, $\overline{P}_4$ and their derivatives are the canonical form of relative invariants. There exist, of course, the relations obtained from $\overline{M} = 0$ and from the differentiation of this equation. The expressions for the invariants in terms of the original coefficients again involve only simple substitutions, use being made of the now known values of ψ and $\bar{\lambda}'$. The questions of the completeness and of the independence of the system are answered precisely as for plane curves.

The expressions $\bar{y}$, $\bar{y}'$, $\bar{y}''$, $\bar{y}'''$ are canonical covariants which may be expressed in terms of the original coefficients and variables by direct substitutions. The covariants thus obtained, together with a complete system of invariants, give a complete system of covariants.

The vertices of the tetrahedron of reference are given by the four fundamental covariants $\bar{y}$, $\bar{y}'$, $\bar{y}''$, $\bar{y}'''$. If $\theta_3 \neq 0$, the plane $x_3 = 0$ is evidently the principal plane* of C and its osculating cubic. The osculating cubic meets $x_3 = 0$ in (1, 0, 0, 0) and also in the point $(3\overline{P}_2' - 8\overline{P}_3, -21\overline{P}_2, 0, -5)$. The osculating plane to the cubic at this point intersects the tangent to C at $\bar{y}$ in the point (0, 1, 0, 0). The totality of osculating planes to the cubic intersects $x_4 = 0$ in lines which envelope a conic, called the osculating conic, whose equation is

$$72\overline{P}_2 x_3^2 - 40x_1x_3 + 15x_2^2 = 0, \quad x_4 = 0.$$

The polar of (0, 1, 0, 0) with respect to this conic is the line $x_2 = 0, x_4 = 0$. The tangent at (0, 1, 0, 0) to the curve generated by $\bar{y}'$ intersects $x_2 = 0$, $x_4 = 0$ in $\bar{y}''$ which is (0, 0, 1, 0), and

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for the plane curve. The equations can be found in Chapter 13 of Wilczynski's book. The expressions for P_i in terms of p_i are of the same form as those for $\overline{P}_i$ in terms of $\bar{p}_i$.

* Halphen, Journal de l'Ecole Polytechnique, vol. 28, p. 25.

the tangent at this point to the curve generated by $\bar{y}''$ intersects $x_3=0$ in $\bar{y}'''$ which is (0, 0, 0, 1). The vertices of the tetrahedron of reference are thus completely determined geometrically. These are essentially the results obtained by Wilczynski and Sannia for the location of the vertices.

The canonical form of (B),

$$\bar{y}^{(iv)} + 6\overline{P}_2\bar{y}'' + 4\overline{P}_3\bar{y}' + \overline{P}_4\bar{y} = 0$$

shows at once the geometrical significance of the vanishing of the fundamental invariants $\overline{P}_2$, $\overline{P}_3$, $\overline{P}_4$. In fact, the curve generated by $\bar{y}'$ is plane if and only if $\overline{P}_4=0$, and the tangent at $\bar{y}'''$ to the curve generated by this point intersects the line joining $\bar{y}$ and $\bar{y}''$ if and only if $\overline{P}_3=0$ and the line joining $\bar{y}$ and $\bar{y}'$ if and only if $\overline{P}_2=0$.

It is easy to make a transformation of coordinates which gives the canonical expansion used by Wilczynski. It is only necessary to change the vertex (0, 0, 1, 0) into the point where the line $x_2=0$, $x_4=0$ intersects the osculating conic and the vertex (0, 0, 0, 1) into the point where the osculating cubic intersects $x_3=0$.

If $\theta_3=0$, the principal plane of C and its osculating cubic is the osculating plane. This fact becomes evident if we observe that the coefficients of ξ^6 are equal in the second expansions of (19) and (22) if $\theta_3=0$. In this case it is easy to verify that the tangent plane at $\bar{y}$ to the quadric osculating C at $\bar{y}$ is the plane $x_3=0$ if

$$\bar{\theta}_4=\overline{P}_4-2\overline{P}_3'+(6/5)\overline{P}_2''-(81/25)\overline{P}_2^2=1.$$

Since $\bar{\theta}_4=\theta_4/(\phi')^4$, we can make $\bar{\theta}_4=1$ by choosing $(\phi')^4=\theta_4$, if $\theta_4\neq 0$. A canonical form for (A) is thus determined in this case and associated with it, just as above, we have two expansions, and a geometrically determined tetrahedron of eference.

If both θ_3 and θ_4 vanish, the curve C is a cubic and has no projective differential properties of interest.

THE UNIVERSITY OF KANSAS

ON THE SEPARATION OF THE PLANE BY A CONTINUUM*

BY R. L. MOORE

In his paper, *On the separation of the plane by irreducible continua*,† W. A. Wilson obtains the following theorem. Let F be the union of two bounded continua H_1 and H_2 having these properties: H_1 and H_2 are irreducible about $A+B$; $H_1 \cdot H_2 = A+B$ where A and B are continua and $A \cdot B = 0$; H_1 and H_2 contain subcontinua C_1 and C_2 respectively such that $\alpha = C_1 \cdot C_2 \cdot A \neq 0$, $\beta = C_1 \cdot C_2 \cdot B \neq 0$, C_1 and C_2 are irreducible between α and β and $F = C_1 + C_2$. Then F cuts the plane and is the frontier of exactly two components of its complement.

In the present paper I will establish two theorems which together yield more information than Wilson's theorem.

THEOREM 1. *If A and B are two mutually exclusive continua and the bounded continua H_1 and H_2 are both irreducible about‡ $A+B$, and $H_1 \cdot H_2 = A+B$, then H_1+H_2 is not the boundary of more than two domains.*

PROOF. Let F denote the point set H_1+H_2. With the aid of the fact that $H_1-(A+B)$ and $H_2-(A+B)$ are§ connected, it may be easily seen that there exists an inversion of the plane about some point of the complement of F such that if, for each point set M, $\overline{M}$ denotes the image of M under this inversion, then no bounded complementary domain of $\overline{A}$ or $\overline{B}$ contains a point of $\overline{F}$. If, to each of the continua $\overline{A}$ and $\overline{B}$, all its bounded complementary domains are added and

* Presented to the Society, December 28, 1927.

† This Bulletin, vol. 33 (1927), pp. 733–744.

‡ If A is a closed subset of a continuum C and no proper subcontinuum of C contains A then C is said to be irreducible about A. See W. A. Wilson, loc. cit., Definition I.

§ See W. A. Wilson, loc. cit., Lemma I.

the continuum so obtained is regarded as an element and each point that does not belong to A or to B is regarded as an element, the collection G of elements so obtained is an upper semi-continuous* collection. Furthermore, no element of the collection G separates the plane and together they fill up the plane. It follows that, if the elements of G are regarded as points, all theorems of the analysis situs of the plane hold true, in the space so obtained, in the sense described, in detail, in my paper *Concerning upper semi-continuous collections of continua.*† Hence, by a theorem used in a similar connection in the proof of Theorem 2, there exists a simple closed curve J of elements of G which separates $\overline{H}_1-(\overline{A}+\overline{B})$ from $\overline{H}_2-(\overline{A}+\overline{B})$. If, for each value of i $(i=1,2)$, t_i' denotes the segment of J whose extremities are the elements of G that contain $\overline{H}_1$ and $\overline{H}_2$, respectively, and D_i denotes the complementary domain of $\overline{F}$ that contains t_i, then, if $\overline{F}$ is the boundary of a domain, that domain must contain a point of t_1 or of t_2, and therefore must be identical with D_1 or with D_2.

THEOREM 2. *If α and β are two mutually exclusive closed point sets and each of the bounded continua C_1 and C_2 is irreducible from* ‡ *α to β, and $C_1 \cdot C_2=\alpha+\beta$, then C_1+C_2 is the boundary of at least two distinct domains.*

PROOF. Let F denote the point set C_1+C_2. With the help of the fact that $C_1-(\alpha+\beta)$ and $C_2-(\alpha+\beta)$ are§ connected

* See my paper, *Concerning upper semi-continuous collections of continua which do not separate a given continuum*, Proceedings of the National Academy, vol. 10 (1924), pp. 356–360.

† Transactions of this Society, vol. 27 (1925), pp. 416–428. In the first line of the statement of Theorem 24 on page 427 of this paper, the word "bounded" should be inserted between the words "closed" and "point".

‡ If K and L are two closed and mutually exclusive point sets, the continuum H is said to be irreducible from K to L if it contains at least one point of K and at least one point of L but contains no proper subcontinuum that does so. See Anna M. Mullikin, *Certain theorems relating to plane connected sets*, Transactions of this Society, vol. 24 (1922), pp. 144–162. Wilson uses the term *irreducible between* instead of *irreducible from*.

§ See Anna M. Mullikin, loc. cit.

it may be shown that there exists an inversion of the plane, about some point of the complement of F, such that, using the notation indicated in the above proof of Theorem 1, no bounded complementary domain of any maximal connected subset of $\bar{\alpha}+\bar{\beta}$ contains any point of $\overline{F}$. To each maximal connected subset X of $\bar{\alpha}$ add all the bounded complementary domains of X and call each point set so obtained an element. Let G_α denote the set of all such elements. Let G_β denote a set of elements determined in the same way from the maximal connected subsets of $\bar{\beta}$. Let G denote the upper semi-continuous collection whose elements are the elements of G_α, the elements of G_β, and the points that belong to no element of G_α or G_β. Let H_1 denote the set of elements $\overline{C}_1-(\bar{\alpha}+\bar{\beta})+G_\alpha+G_\beta$ and let H_2 denote the set $\overline{C}_2-(\bar{\alpha}+\bar{\beta})+G_\alpha+G_\beta$. The set of elements $G_\alpha+G_\beta$ is a totally disconnected set of elements and it consists of all elements common to the two closed, connected and bounded sets H_1 and H_2. Let Q denote a simple closed curve enclosing H_1+H_2. There exists an arc AB such that (1) A belongs to Q, (2) B belongs to $\overline{C}_1-(\bar{\alpha}+\bar{\beta})$ or to $\overline{C}_2-(\bar{\alpha}+\bar{\beta})$ and (3) B is the only element of AB that belongs to H_1+H_2. Suppose B belongs to $\overline{C}_1-(\bar{\alpha}+\bar{\beta})$. Then $\overline{C}_2-(\bar{\alpha}+\bar{\beta})$ lies in a bounded complementary domain of $Q+AB+H_1$. Call this domain D. There exists* a simple closed curve J of elements of G such that (1) J contains at least one element of $G_\alpha+G_\beta$ and encloses at least one element of $\overline{C}_2-(\bar{\alpha}+\bar{\beta})$, (2) $(J+G_\alpha+G_\beta)-(G_\alpha+G_\beta)$ is a subset of D and it contains no element of $\overline{C}_2-(\bar{\alpha}+\bar{\beta})$. With the help of the fact that $\overline{C}_2-(\bar{\alpha}+\bar{\beta})$ is connected and that every element of $G_\alpha+G_\beta$ is a limit element of $\overline{C}_2-(\bar{\alpha}+\bar{\beta})$ it easily follows that J contains $G_\alpha+G_\beta$ and encloses the whole of $\overline{C}_2-(\bar{\alpha}+\bar{\beta})$. Furthermore, $\overline{C}_1-(\bar{\alpha}+\bar{\beta})$ is wholly without J. It is easy to see that the curve J contains two segments t_1 and t_2 such that (1) for each i $(i=1, 2)$ t_i contains no element of G_α or of G_β but the end elements of t_i belong to G_α and to G_β respectively and

* See this Bulletin, vol. 33 (1927), p. 521, Abstract 38.

(2) t_1 and t_2 belong to different complementary domains of H_1+H_2. The segments t_1 and t_2 are point sets in the ordinary sense* and they are connected subsets of the complement (in the ordinary sense) of $\overline{F}$ and each of the sets $\bar{\alpha}$ and $\bar{\beta}$ contains at least one limit point of t_1 and at least one limit point of t_2. In the space of the ordinary points of the plane, for each i let D_i denote the complementary domain of $\overline{F}$ that contains t_i. The continuum $\overline{F}$ is the boundary both of D_1 and of D_2. For let N denote the boundary of D_1. If N does not contain $\overline{C}_2$ then $N \cdot \overline{C}_2$ contains no connected subset containing both a point of some element of G_1 and a point of some element of G_2. Hence there exists an arc A_1A_2 from a point A_1 of t_1 to a point A_2 of t_2 and lying, except for these points, wholly in the interior of J and having no point in common with $N \cdot \overline{C}_2$. Since $\overline{C}_1-(\bar{\alpha}+\bar{\beta})$ lies wholly without J it has no point in common with A_1A_2. Thus A_1A_2 has no point in common with N. But A_1 and A_2 belong to different complementary domains of N. Thus the supposition that N does not contain $\overline{C}_2$ has led to a contradiction. In a similar way it may be shown that N contains $\overline{C}_1$. Therefore $\overline{F}$ is the boundary of D_1. By a similar argument it may be shown to be the boundary of D_2. It follows that F is the boundary of at least two domains.

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* This does not imply that they are necessarily subsets of *arcs* whose elements are all points in the ordinary sense. It is possible, for example, that t_1 may be the set of all points of the graph of $y=\sin(1/x)$ that lie between the lines $x=0$ and $x=1$. This point set is not a subset of any arc whose elements are all points but it is a segment of an arc of which the only element which is not a point is the straight line interval from (0,1) to (0,−1).

A CHARACTERISTIC PROPERTY OF A CO-SET

BY G. A. MILLER

In view of the fundamental importance of co-sets in the theory of groups it may be of interest to note here the following characteristic property of a co-set. *A necessary and sufficient condition that a set of group operators* $s_1, s_2, \cdots, s_l$, *which is not itself a group be a right co-set is that it involve at least one operator such that if the inverse of this operator is multiplied on the right into every operator of the set the resulting products constitute a group.* In this theorem the word right may be replaced by the word left. A proof of this theorem results directly from the fact that if the two sets

$$s_1 s_\alpha^{-1}, s_2 s_\alpha^{-1}, \cdots, s_l s_\alpha^{-1}, \quad \text{and} \quad t_1, t_2, \cdots, t_l,$$

where α is one of the numbers $1, 2, \cdots, l$, are identical and constitute a group, then the latter set multiplied on the right by s_α is a right co-set. It may be noted that if the former of these sets is a group for one value of α it must be the same group for every one of these values.

It should also be noted that if the operators $s_1, s_2, \cdots, s_l$ satisfy the condition that when every one of them is multiplied on the right by the inverse of one of them, the resulting operators constitute a group, then they must also constitute a group when every one of them is multiplied on the left by the inverse of one of them, and vice versa. A necessary and sufficient condition that these two groups be the same is that at least one of the operators $s_1, s_2, \cdots, s_l$ transform the group $t_1, t_2, \cdots, t_l$ into itself, and if this condition is satisfied then this group is transformed into itself by every one of the operators $s_1, s_2, \cdots, s_l$. It must therefore be an invariant subgroup of the group generated by these operators.

If the set of operators $s_1, s_2, \cdots, s_l$ has the property that $s_1 s_\alpha, s_2 s_\alpha, \cdots, s_l s_\alpha$ is a group for a certain value of α then

s_α^{-1} must be found in this set. Hence it results from what was said above that the set is a right co-set which involves the inverse of one of its operators. It does not follow, however, that $s_1 s_\alpha, s_2 s_\alpha, \cdots, s_l s_\alpha$ is a group for every value of α from 1 to l if it is a group for some one of these values. If it is a group for every one of these values then this group must be a subgroup of index 2 under the group generated by $s_1, s_2, \cdots, s_l$. Moreover, this set must include the inverse of every one of its operators. It is obvious that similar remarks apply to the set if $s_\alpha s_1, s_\alpha s_2, \cdots, s_\alpha s_l$ constitute a group.

If a set of group operators $s_1, s_2, \cdots, s_l$ has the property that it as a whole is unchanged when it is multiplied on the right by an operator t which is not necessarily contained in the set, then it is composed either of co-sets or of augmented co-sets with respect to the group generated by t. If it is also unchanged when it is multiplied on the right by an operator t' which is not generated by t, then it must be composed either of co-sets or of augmented co-sets with respect to the group generated by t and t'. In general, if the set is unchanged as a whole when it is multiplied on the same side successively by various operators then it must be composed either of co-sets or of augmented co-sets with respect to the group generated by these multipliers. In particular, l mus tbe divisible by the order of the group generated by these multipliers. It should be noted that this is an extension of Lagrange's fundamental theorem, since in the case when $s_1, s_2, \cdots, s_l$ is a group it is obvious that it as a whole is unchanged when it is multiplied by the generators of an arbitrary subgroup, and hence the order of a group must be divisible by the order of each of its subgroups according to the more general theorem just noted.

When the multiplying operators are contained in the set $s_1, s_2, \cdots, s_l$ it results from the preceding paragraph that the set includes the group generated by the multipliers. If it is not a group itself it must therefore be composed of augmented co-sets with respect to the group generated by these

multipliers. It may be noted that these considerations are frequently useful in determining whether a given set of operators constitutes a group when we can readily find whether the set involves the group generated by certain of its operators. The fact that a set of operators which involves the square of every one of them and also the product of every pair of them does not necessarily constitute a group results directly from operators involved in the symmetric group of order 6. It is necessary to add that it involves these products when the factors are taken in both of the possible orders.

From what precedes it results that *if a set of operators has the property that it as a whole remains unchanged when all the operators of this set are multiplied on the same side by various operators, then it either contains all the operators of the group generated by these multipliers or it contains no operator of this group.* In the former case it is either this group or it consists of augmented co-sets with respect to it, while in the latter case it must be composed of one or more co-sets with respect to the same group. Hence, a characteristic property of co-sets and of augmented co-sets is that the totality of the operators involved therein remains unchanged when all of these operators are multiplied on the same side by a given operator. If such a totality is composed of right co-sets or augmented right co-sets it is not necessarily also composed of left co-sets or of augmented left co-sets.

THE UNIVERSITY OF ILLINOIS

A NEW NORMAL FORM FOR QUARTIC EQUATIONS*

BY RAYMOND GARVER

1. *Introduction.* There are a number of well known normal forms to which the general quartic equation may be reduced by means of a Tschirnhaus transformation, without requiring the solution of any equation of higher than the third degree. The reduction to $y^4+c_3y+c_4=0$ or to $y^4+c_2y^2+c_4=0$ requires no comment. Bring first showed how to obtain the form $y^4+c_1y^3+c_4=0$, by applying a simple reciprocal transformation (which is reducible to a Tschirnhaus transformation) to the first form above.† The reduction to the binomial form $y^4+c_4=0$ by an ordinary third degree transformation leads to a sixth degree equation, but Lagrange was able to show (by an a priori proof) that this sextic will factor into three quadratics whose coefficients are themselves roots of cubic equations.‡ The required factorization would, however, be difficult actually to obtain; the necessary transformation may be obtained more conveniently in another way.§

I wish in this paper to discuss a new normal form, $y^4+c_2y^2=0$. I first give an a priori proof, based on that of Lagrange for the transformation to binomial form, that the reduction does not involve the solution of any equation of higher than the third degree. Then, using a theorem of Hermite and certain results of Cayley, I consider the setting up of the necessary transformation.

2. *The a Priori Proof.* If we apply the transformation

$$y = x^3 + k_2x^2 + k_3x + k_4 \tag{1}$$

to the general quartic equation

* Presented to the Society, April 7, 1928.

† Bring's work was published in 1786. It is reproduced in Grunert's Archiv, vol. 41 (1864), pp. 105–112.

‡ See his *Oeuvres*, vol. III, Paris, 1869, pp. 284–295. His result was originally published in 1770.

§ See this Bulletin, vol. 33 (1927), No. 6.

$$x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0, \tag{2}$$

the conditions that the transformed equation in y lack its second, fourth and fifth terms are, as can easily be verified by making use of Newton's relations,

$$\sum y = 0,\ \sum y^3 = 0,\ (\sum y^2)^2 - 2\sum y^4 = 0. \tag{3}$$

But the first of these equations is linear in k_2, k_3, k_4, the second is of the third degree, while the third is of the fourth degree. Hence if we eliminate two of the k_i, say k_3 and k_4, between these equations, we shall expect to be led to a final equation of the twelfth degree in k_2.

The problem is to show that k_2 can be found without requiring the solution of any equation of higher than the third degree. To do this, it is necessary to express k_2 as a function of the roots x_1, x_2, x_3, x_4 of the given quartic, on the assumption that the transformed equation is $y^4+c_2y^2=0$. The roots of this latter equation are 0, 0, and, let us say, r and $-r$. We must then have, from (1),

$$\left\{\begin{aligned} 0 &= x_1^3 + k_2x_1^2 + k_3x_1 + k_4, \\ 0 &= x_2^3 + k_2x_2^2 + k_3x_2 + k_4, \\ r &= x_3^3 + k_2x_3^2 + k_3x_3 + k_4, \\ -r &= x_4^3 + k_2x_4^2 + k_3x_4 + k_4. \end{aligned}\right. \tag{4}$$

It is now a matter of simple algebra to solve these equations for k_2; the result may be written in the form

$$k_2 = -\frac{(x_1^3+x_2^3-x_3^3-x_4^3)(x_1-x_2)-(x_1^3-x_2^3)(x_1+x_2-x_3-x_4)}{(x_1^2+x_2^2-x_3^2-x_4^2)(x_1-x_2)-(x_1^2-x_2^2)(x_1+x_2-x_3-x_4)}. \tag{5}$$

We may now find the number of different values which k_2 can assume by making the 24 possible permutations on the roots x_1, x_2, x_3, x_4. If the reader goes through this work, he will find that k_2 can take on only six different values, $-M/P$, $-M'/P'$, $-M''/P''$, $-N/Q$, $-N'/Q'$, $-N''/Q''$, where M, N, P, Q have the values

$$(6)\quad \begin{cases} M = (x_1^3 + x_2^3 - x_3^3 - x_4^3)(x_1 - x_2) \\ \qquad\qquad - (x_1^3 - x_2^3)(x_1 + x_2 - x_3 - x_4), \\ N = (x_1^3 + x_2^3 - x_3^3 - x_4^3)(x_3 - x_4) \\ \qquad\qquad - (x_3^3 - x_4^3)(x_1 + x_2 - x_3 - x_4), \\ P = (x_1^2 + x_2^2 - x_3^2 - x_4^2)(x_1 - x_2) \\ \qquad\qquad - (x_1^2 - x_2^2)(x_1 + x_2 - x_3 - x_4), \\ Q = (x_1^2 + x_2^2 - x_3^2 - x_4^2)(x_3 - x_4) \\ \qquad\qquad - (x_3^2 - x_4^2)(x_1 + x_2 - x_3 - x_4), \end{cases}$$

where M', N', P', Q' are obtained from M, N, P, Q by interchange of the subscripts 2 and 3, and where M'', N'', P'', Q'' are obtained from M, N, P, Q by replacing the subscripts 2, 3, 4 by 4, 2, 3, respectively. We thus see that k_2 can depend only on an equation of the sixth degree; that is, the twelfth degree equation mentioned above must, in this case, reduce to one of the sixth degree.

We now wish to go further, and show that this sixth degree equation can be broken up into quadratic factors, whose coefficients are themselves roots of equations of degree not higher than the third. Consider the quadratic equation $k_2^2 + tk_2 + u = 0$, whose roots are $-M/P$ and $-N/Q$, and whose left-hand member is therefore a priori a factor of the left-hand member of the sextic. We have

$$(7)\qquad t = \frac{M}{P} + \frac{N}{Q} = \frac{MQ + NP}{PQ}, \qquad u = \frac{MN}{PQ}.$$

It can now be verified that t and u each take on only three different values under the 24 permutations on the roots x_1, x_2, x_3, x_4. These values are

$$(8)\quad \begin{cases} t_1 = \dfrac{MQ + NP}{PQ}, & u_1 = \dfrac{MN}{PQ}, \\ t_2 = \dfrac{M'Q' + N'P'}{P'Q'}, & u_2 = \dfrac{M'N'}{P'Q'}, \\ t_3 = \dfrac{M''Q'' + N''P''}{P''Q''}, & u_3 = \dfrac{M''N''}{P''Q''}. \end{cases}$$

The equation in t having the roots t_1, t_2, t_3 has, moreover, coefficients which are rational functions of the coefficients of the given equation; this follows because the elementary symmetric functions of the t's are symmetric functions of the x's, as can again be verified easily. A similar statement can be made concerning the cubic equation in u having the roots u_1, u_2, u_3.

The sextic in k_2 thus has a quadratic factor whose coefficients are roots of cubic equations; in fact it clearly has three such factors and may be written

$$(9) \quad (k_2^2 + t_1 k_2 + u_1)(k_2^2 + t_2 k_2 + u_2)(k_2^2 + t_{3\ 2} + u_3) = 0.$$

This completes, except for certain details which can be filled in easily enough, the a priori demonstration that the general quartic equation can be reduced to the form $y^4 + c_2 y^2 = 0$ without solving any equation of higher degree than the third. This demonstration does not, however, lead to a convenient determination of the necessary transformation.

3. *The Transformation.* The use of a special form of the Tschirnhaus transformation introduced by Hermite makes the computation of the transformed equation in y relatively simple.* This depends on certain invariantive properties which need not be stated explicitly; I shall merely say that his transformation for the quartic is of the form

$$(10) \quad y = (ax + b)B + (ax^2 + 4bx + 3c)C + (ax^3 + 4bx^2 + 6cx + 3d)D,$$

where B, C and D are the parameters, and where the original quartic is

$$(11) \qquad ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0.$$

Cayley† has computed the transformed quartic, as well as its invariants of the second and third degree. I make use of

* Comptes Rendus, vol. 46 (1858), p. 961; *Oeuvres*, vol. 2, Paris, 1908, pp. 30–37.

† *Collected Papers*, vol. 4, Cambridge, 1891, pp. 375–394; vol. 5, 1892, 449–453.

these last two expressions. If we use S and T to represent the quadratic invariant $ae-4bd+3c^2$ and the cubic invariant $ace+2bcd-ad^2-eb^2-c^3$ of (11), and if we put

$$(12)\quad \begin{cases} \phi = aB^2 + 4bBC + c(2BD + 4C^2) + 4dCD + eD^2, \\ \Delta = \dfrac{2}{3}(BD - C^2), \end{cases}$$

then the corresponding invariants of the transformed equation, which we may denote by S_y and T_y, are

$$(13)\quad \begin{cases} S_y = S\phi^2 + 18T\phi\Delta + 3S^2\Delta^2, \\ T_y = T\phi^3 + S^2\phi^2\Delta + 9ST\phi\Delta^2 + (54T^2 - S^3)\Delta^3. \end{cases}$$

I first show that the parameters B, C and D can be chosen so that the transformed equation will have $S_y^3-27T_y^2$ equal to zero. This expression is clearly of the sixth degree in ϕ and Δ, but it reduces to (S^3-27T^2) multiplied by the square of $(\phi^3-9S\phi\Delta^2-54T\Delta^3)$. Hence the condition $S_y^3-27T_y^2=0$ is a cubic in ϕ/Δ with known coefficients. If a root of this cubic is r, we have then to consider the equation $\phi=r\Delta$, which is a homogeneous, quadratic equation in B, C and D. One of these can be taken equal to zero, and the equation solved for the ratio of the other two.

Now it is well known that any quartic for which S^3-27T^2 is equal to zero can be reduced by a linear fractional transformation (which is reducible to a Tschirnhaus transformation) to the normal form $y^4+c_2y^2=0$. In fact it is not difficult to show that this reduction requires only the solution of quadratic equations, but I shall not reproduce the details of this calculation. Finally, since the product of two Tschirnhaus transformations is itself a Tschirnhaus transformation, we arrive at the desired result.

THE UNIVERSITY OF ROCHESTER

EXISTENCE THEOREMS FOR IMPLICIT FUNCTIONS OF REAL VARIABLES*

BY H. J. ETTLINGER

The classical theorems on implicit functions make use of the continuity of the given functions and their partial derivatives when all the variables are considered as independent.† The existence theorems established herein bring the implicit function theorems into line with the most recent developments of real variable theory.‡ Two of my students, W. M. Whyburn and J. H. Sturdivant, have made use of Theorem I in connection with studies of the properties of solutions of ordinary linear differential equations with summable coefficients.

In Theorem I sufficient conditions are given to ensure a single-valued continuous solution $y=y(x)$ of the relation $F(x, y)=0$. These conditions reduce the classical conditions considerably.

By introducing symmetry in x and y save in the final two hypotheses of Theorem II, sufficient conditions are given to ensure a single-valued absolutely continuous solution (with a summable derivative almost everywhere).

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* Presented to the Society, September 8, 1926.

† See Goursat-Hedrick, *Mathematical Analysis*, vol. 1, 1904, Chapter II p. 35ff. For a summary of the results and references to original sources, see Bliss, Princeton Colloquium Lectures, *Fundamental Existence Theorems*, delivered in 1909, published by the Society in 1913, New York.

A distinct lightening of the classical conditions for the existence of implicit functions is to be found in a paper by Hedrick and Westfall, Bulletin de la Société de France, vol. 44 (1916), pp. 1–14.

‡ See Carathéodory, *Vorlesungen über reelle Funktionen*, Leipzig, Teubner, 1918. Very recently (August, 1927) a second edition of this treatise has appeared with only a few additional references to more recent literature. The page references given in later footnotes to the present work apply equally well to the second (1927) edition.

THEOREM I. HYPOTHESIS. 1. *D_h is a neighborhood of a point (X, Y) in the (x, y) plane, $|x-X| \leqq h$, $|y-Y| \leqq h$, $h>0$.* 2. *$F(x, y)$ is defined in D_h and vanishes at (X, Y).* 3. *(a) $F(x, y)$ is absolutely continuous in y on $|x-X| \leqq h$, for every fixed x on $|x-X| \leqq h$; (b) $|F_y'(x, y)| \leqq M(y)$ for all values of x on $|x-X| \leqq h$, where $M(y)$ is summable in y on $|y-Y| \leqq h$; (c) $F_y'(x, y)$ is continuous in x for every fixed y almost everywhere on $|y-Y| \leqq h$.* 4. *$F(x, Y)$ is continuous in x on $|x-X| \leqq h$.* 5. *$F_y'(x, y)>0$ (<0)* for each fixed x on $|x-X| \leqq h$, almost everywhere on $|y-Y| \leqq h$.*

CONCLUSION. *There exists in D_k, $0<k \leqq h$, a unique single-valued continuous function $y=y(x)$, such that we have* 1. *$F(x, y(x)) \equiv 0$ in D_k,* 2. *$Y=y(X)$,* 3. *$y(x)$ is continuous on $|x-X| \leqq k$.*

PROOF. By a theorem due to Carathéodory† it follows that $F(x, y)$ is continuous in (x, y) at (x, Y) for $|x-X| \leqq h$ and

$$F(x, y) = F(x, Y) + \int_Y^y F_t'(x, t)dt.$$

From hypothesis 5, it follows that for a fixed x, $F(x, y)$ is a monotonic increasing (decreasing) function of y on $|y-Y| \leqq h$, and since $F(X, Y)=0$,

$$F(X, y) = \int_Y^y F_t'(X, t)dt.$$

Hence $F(X, y)<0$ for $y<Y$, and $F(X, y)>0$ for $y>Y$, or $F(X, Y-h)<0$ and $F(X, Y+h)>0$.

Since $F(x, y)$ is continuous in x, it follows that there exists a neighborhood of (X, Y) contained in D_h such that $F(x, Y-k)<0$ and $F(x, Y+k)>0$ for every x on $|x-X| \leqq k$.

* For the case (<0) all the subsequent inequality signs are reversed.

† Loc. cit., p. 678, Satz 5. The theorem made use of here is not explicitly stated by Carathéodory, but is implicit in the existence theorem for differential equations cited above. See my note, *On continuity in several variables*, this Bulletin, vol. 33 (1927), p. 37. Hypotheses 3 (*a*) and 3 (*b*) of Theorem I above should replace hypotheses (2) and (3) of the theorem of my note.

But for any fixed x on $|x-X| \leqq k$, $F(x, y)$ is monotonic increasing in y. Hence for each x on $|x-X| \leqq k$, there is one and only one value of y, $y=y(x)$, on $|y-Y| \leqq k$, such that $F(x), y(x) \equiv 0$. Now for $x=X$, we see that the corresponding value of y is Y. Finally by the very method* of obtaining $y(x)$, we see that $y(x)$ is continuous in x on $|x-X| \leqq k$.

THEOREM II. HYPOTHESIS. 1, 2, 3, 5 *remain as in Theorem* I. 4.(*a*) $F(x, y)$ *is absolutely continuous*† *in* x *on* $|x-X| \leqq h$, *for every fixed* y *on* $|y-Y| \leqq h$; (*b*) $|F_x'(x, y)| \leqq N(x)$ *for all values of* y *on* $|y-Y| \leqq h$, *where* $N(x)$ *is summable in* x *on* $|x-X| \leqq h$; (*c*) $F_x'(x, y)$ *is continuous in* y *for every fixed* x *almost everywhere on* $|x-X| \leqq h$. 6. $|F_x'(x, y)/F_y'(x, y)| \leqq K(x)$ *for every* y *on* $|y-Y| \leqq h$, *where* $K(x)$ *is summable in* x *on* $|x-X| \leqq h$.

CONCLUSION. 1. *There is one and only one solution of Theorem* I, $y=y(x)$, *which is absolutely continuous in* x *on* $|x-X| \leqq k$, *and* 2. $y_x' = -F_x'(x, y)/F_y'(x, y)$ *almost everywhere on* $|x-X| \leqq k$, *where* y_x' *is summable on* $|x-X| \leqq k$.

PROOF. Condition 4 of Theorem II carries with it condition 4 of Theorem I. From Theorem I we have a solution $y=y(x)$, such that $F(x, y(x)) \equiv 0$. Let (x, y) and $(x+\Delta x, y+\Delta y)$ be any two points on $F(x, y(x)) \equiv 0$. We may write the identity

$$\frac{\Delta y}{\Delta x} = -\frac{\dfrac{\displaystyle\int_x^{x+\Delta x} F_t'(t, y+\Delta y)\,dt}{\Delta x}}{\dfrac{\displaystyle\int_y^{y+\Delta y} F_t'(x, t)\,dt}{\Delta y}} \tag{1}$$

* This follows exactly as in the classical theorem, see Goursat-Hedrick, loc. cit., p. 37.

† Hypothesis 4 is exactly symmetric to hypothesis 3 with respect to x and y.

The right side of (1) has the limit*

$$-\frac{F_z'(x, y(x))}{F_y'(x, y(x))}, \tag{2}$$

except for a null set of x values on $|x-X| \leqq k$. Hence the left side of (1) approaches a limit almost everywhere on $|x-X| \leqq k$, or

$$y_x' = -\frac{F_x'(x, y(x))}{F_y'(x, y(x))}. \tag{3}$$

By a theorem due to Carathéodory† the numerator and denominator of (2) are measurable functions of x on $|x-X| \leqq k$, and hence summable by hypothesis 6. Hence by (3) $y(x)$ is absolutely continuous in x on $|x-X| \leqq k$.

The above theorems may be extended by the usual method of induction to a system of n functions in n dependent variables and m independent variables.‡

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* This follows from a generalization of Theorem III of my review, *Schlesinger on Lebesgue integrals*, this Bulletin, vol. 33 (1927) p. 111. A detailed proof is given by W. M. Whyburn in his dissertation as yet unpublished (offered to the Transactions of this Society).

† Loc. cit., p. 665, Satz 1.

‡ See Goursat-Hedrick, loc. cit., p. 45, etc.

THE CONVERGENCE OF DOUBLE FOURIER SERIES OF A CERTAIN TYPE*

BY G. M. MERRIMAN†

Occasion has arisen recently, in connection with double Fourier series whose coefficients satisfy certain conditions, for the development of a necessary condition for the convergence of such series. The present paper obtains such a condition, following a generalization to two variables of a theorem of Fatou.‡

Let $f(\alpha, \beta)$ be a periodic summable function whose double Fourier series, supposed for convenience to contain no terms independent of α or β, is given by

$$f(\alpha,\beta) \sim \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\begin{array}{l} a_{m,n}\cos m\alpha\cos n\beta + b_{m,n}\cos m\alpha\sin n\beta \\ + c_{m,n}\sin m\alpha\cos n\beta + d_{m,n}\sin m\alpha\sin n\beta \end{array}\right\},$$

which will be abbreviated as

$$(1) \qquad f(\alpha,\beta) \sim \sum_{1}^{\infty}\sum_{1}^{\infty}(a,b,c,d,\alpha,\beta)_{m,n} = \sum_{1}^{\infty}\sum_{1}^{\infty}A_{m,n}.$$

The integral of $f(\alpha, \beta)$, obtained by integrating (1) once with respect to each variable, is§

$$(2) \qquad g(\alpha,\beta) = \sum_{1}^{\infty}\sum_{1}^{\infty}\frac{1}{mn}(d, -c, -b, a, \alpha, \beta)_{m,n}.$$

We wish to prove the following theorem.

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‡ P. Fatou, *Séries trigonométriques et séries de Taylor*, Acta Mathematica, vol. 30 (1906), p. 385.

§ For the equality sign, see W. H. Young, *Multiple Fourier series*, Proceedings of the London Mathematical Society, (2), vol. 11, §§10 and 13.

THEOREM. *If $a_{m,n}$, $b_{m,n}$, $c_{m,n}$, and $d_{m,n}$ are $o(1/mn)$, and if $A^{-1,0}_{m,n}$ and $A^{0,-1}_{m,n}$ are $o(1/m)$ and $o(1/n)$ respectively,* then a necessary condition that* (1) *should be convergent is the existence of*

$$(3)\quad \lim_{x,y\to 0}\left\{\frac{1}{2xy}\left[g(\alpha+x,\beta+y)+g(\alpha-x,\beta-y)-g(\alpha+x,\beta)\right.\right.$$
$$\left.\left.-g(\alpha-x,\beta)-g(\alpha,\beta+y)-g(\alpha,\beta-y)+2g(\alpha,\beta)\right]\right\}=f(\alpha,\beta).$$

We first prove the following lemma.

LEMMA. *If* (1) *is convergent and its coefficients satisfy the conditions of the theorem, then there exists*

$$(4)\quad \lim_{x,y\to 0}\left\{\frac{1}{4xy}\left[g(\alpha+x,\beta+y)-g(\alpha+x,\beta-y)-g(\alpha-x,\beta+y)\right.\right.$$
$$\left.\left.+\,g(\alpha-x,\beta-y)\right]\right\}=f(\alpha,\beta).$$

We replace the fractional expression in (4) by

$$(5)\qquad \sum_1^\infty\sum_1^\infty A_{m,n}\,\frac{\sin mx}{mx}\,\frac{\sin ny}{ny},$$

and we desire to show that this expression converges to $f(\alpha,\beta)$ as $x, y\to 0$. Our procedure is to divide the sum in (5) into four sums, each one to be considered separately:

$$(6)\quad \sum_1^\infty\sum_1^\infty=\sum_{r+1}^\infty\sum_{s+1}^\infty+\sum_{r+1}^\infty\sum_1^s+\sum_1^r\sum_{s+1}^\infty+\sum_1^r\sum_1^s,$$

where r and s are the largest integers less than π/x and π/y respectively.

(i) Consider the first sum in (6). By hypothesis we have that $A_{m,n}$ is $o[1/(mn)]$, that is, for $m>M, n>N, mn\,|A_{m,n}|<\epsilon$, ϵ having been previously assigned. Hence, for $r>M$, $s>N$,

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* For convenience of notation, we put $A^{-1,-1}_{m,n}=A_{m,n}$, $A^{-1,0}_{m,n}=\sum_{n-1}^n A^{-1,-1}_{m,n}$, $A^{0,-1}_{m,n}=\sum_{m-1}^m A^{-1,-1}_{m,n}$, $A^{0,0}_{m,n}=\sum_1^m\sum_1^n A^{-1,-1}_{m,n}$.

$$\left| \sum_{r+1}^{\infty} \sum_{s+1}^{\infty} A_{m,n} \frac{\sin mx}{mx} \frac{\sin ny}{ny} \right| < \frac{\epsilon}{xy} \sum_{r+1}^{\infty} \sum_{s+1}^{\infty} \frac{1}{m^2 n^2}$$

$$< \frac{\epsilon}{rsxy} < \frac{\epsilon}{(\pi - x)(\pi - y)} < \frac{\epsilon}{9}$$

if $x, y < .1$.

(ii) For the consideration of the second sum of (6) (and that of the third is wholly analogous to the analysis to be given here), we note that by the hypothesis $A_{m,n}^{-1,0}$ is $o(1/m)$, that is, $m \left| \sum_{n=1}^{n} A_{m,n} \right| < \epsilon$, for $m > M$. Again, $(\sin ny)/(ny)$ is a monotonically decreasing function of increasing n, and is always positive and less than unity since ny is between 0 and π. Therefore, by Abel's lemma,

$$\left| \sum_{r+1}^{\infty} \sum_{1}^{s} A_{m,n} \frac{\sin mx}{mx} \frac{\sin ny}{ny} \right| < \sum_{r+1}^{\infty} \frac{\epsilon}{m} \frac{|\sin mx|}{mx}$$

$$< \frac{\epsilon}{x} \sum_{r+1}^{\infty} \frac{1}{m^2} < \frac{\epsilon}{\pi - x} < \frac{\epsilon}{3}$$

if $x < .1$. Similarly,

$$\left| \sum_{1}^{r} \sum_{s+1}^{\infty} A_{m,n} \frac{\sin mx}{mx} \frac{\sin ny}{ny} \right| < \frac{\epsilon}{3}$$

if $y < .1$.

(iii) We divide the fourth sum of (6) into two parts: $\sum_1^M \sum_1^N$ and

$$\left\{ \sum_{1}^{M} \sum_{N+1}^{s} + \sum_{M+1}^{r} \sum_{1}^{N} + \sum_{M+1}^{r} \sum_{N+1}^{s} \right\}.$$

In the second sum, we use the fact that $(\sin mx)/(mx)$ and $(\sin ny)/(ny)$ are both monotonically decreasing and are less in absolute value than unity, and the fact that the convergence of $\sum\sum A_{m,n}$ implies that, for M and N large enough,

$$\left| \left\{ \sum_{1}^{M} \sum_{N+1}^{s} + \sum_{M+1}^{r} \sum_{1}^{N} + \sum_{M+1}^{r} \sum_{N+1}^{s} \right\} A_{m,n} \right|$$

can be made less than $2\epsilon/9$, whatever the values of r and s.*

* See T. J. I'A. Bromwich, *Infinite Series*, §29 and §37, the latter containing the generalization of Abel's lemma which we have used.

In the first sum we note that as $x, y \to 0$, the sum approaches $\sum_1^M \sum_1^N A_{m,n}$.

Collecting the results of (i), (ii), and (iii), we find that as $x, y \to 0$, (5) converges to $f(\alpha, \beta)$, completing the lemma.

To prove the theorem we have only to combine (4) with

$$(7)\quad \lim_{x,y\to 0}\left\{\frac{1}{xy}[g(\alpha+x,\beta+y)+g(\alpha+x,\beta-y)+g(\alpha-x,\beta+y) +g(\alpha-x,\beta-y)+4g(\alpha,\beta)-2g(\alpha+x,\beta)-2g(\alpha-x,\beta) - 2g(\alpha,\beta+y)-2g(\alpha,\beta-y)]\right\} = 0,$$

which is a generalization to two variables of a result due to Riemann*.

Instead of requiring the existence of the limit (3), we can replace that condition by the requirement of the existence of (4), as is shown by our lemma. But the requirement of the existence of (4) immediately translates into that of the existence of

$$\frac{1}{4xy}\int_{-x}^{x}\int_{-y}^{y} f(\alpha+u,\ \beta+v)dudv \to f(\alpha,\ \beta)$$

as $x, y \to 0$, which in turn is equivalent to requiring the existence of

$$(8)\quad \frac{1}{4xy}\int_{0}^{x}\int_{0}^{y}[f(\alpha+u,\ \beta+v)+f(\alpha-u,\ \beta+v) + f(\alpha+u,\ \beta-v)+f(\alpha-u,\ \beta-v)]dudv \to f(\alpha,\ \beta)$$

as $x, y \to 0$. Hence, we have the following corollary.

COROLLARY. *If the conditions of the theorem are satisfied, then a necessary condition for the convergence of* (1) *is the existence of* (8).

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* See Hilda Geiringer, *Trigonometrische Doppelreihen*, Monatshefte für Mathematik und Physik, 1918, p. 73.

TERNARY CHARACTERISTICS OF PRIMES*

BY E. T. BELL

1. *Introduction.* As will be seen from the proofs, the curious properties of primes given by the theorems of §2 are less abstruse than they seem at first sight. That they are obvious when once the clue to their derivation is given does not, however, detract from their interest.

Let μ be either of 1, 2, and λ a definite one of 0, 1. We shall denote the equation

$$x^2 - a[\lambda + (1-\lambda)p^\alpha]y^2 - bq^\beta z^2 = \mu^2 \tag{1}$$

by its *characteristic* $[a, b, \lambda]$. Thus $[a, b, 0]$ is

$$x^2 - ap^\alpha y^2 - bq^\beta z^2 = \mu^2, \tag{2}$$

and $[a, b, 1]$ is

$$x^2 - ay^2 - bq^\beta z^2 = \mu^2. \tag{3}$$

To avoid separate statements, we do not distinguish the cases $[a, b, \lambda]$, $\lambda = 0, 1$, until necessary. In (1), $p, q, \alpha, \beta, x, y, z$ are variable integers subject only to the following restrictions: p, q are primes >2; $\alpha > 0$, $\beta > 0$ are $\equiv 1 \pmod 4$; $x > 0$, $y > 0$, $z > 0$; a, b are constant integers >0. If, subject to all these restrictions,

$$p = p', \quad q = q', \quad \alpha = \alpha', \quad \beta = \beta', \quad x = x', \quad y = y', \quad z = z'$$

is a set of values of $p, q, \alpha, \beta, x, y, z$ for which (1) is true, we shall call the matrix $(p', q', \alpha', \beta', x', y', z')$ a *solution* of (1). Note in particular that the definitions imply that p', q' in each solution are primes >2. The equation $[a, b, \lambda]$ is transcendental; α, β are variables. When $\lambda = 1$ we may omit p, α, p', α' from the definitions, since their retention is then trivial; a solution of $[a, b, 1]$ may be written $(*, q', *, \beta',$

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x', y', z'). Solutions being matrices, their equality is defined. Unequal solutions will be called *distinct*.

If p', q' in a given solution are such that, for m constant and >2,

$$p' \equiv p_0', \quad q' \equiv q_0', \mod m, \quad p_0' > 0, \quad q_0' > 0,$$

we shall say that the solution has the *residue* $(p_0', q_0')_m$ *modulo* m.

A set consisting of an infinity of distinct solutions is said to be *infinite*. Infinite sets are called *distinct* when each of the sets contains an infinity of solutions not in any of the others.

An integer $D>0$ which determines the constants a, b, and λ, so that (1) has an infinite set of solutions will be called a *determinant* of $[a, b, \lambda]$. The precise way in which D determines $[a, b, \lambda]$ so as to have the stated property is immaterial for the moment.

Let c, n, r_j, s be constant integers, $c>0$, $n>2$, $r_j>0$ $(j=1, \cdots, s)$. Let the δ_j $(j=1, \cdots, s)$ be distinct primes >2 such that δ_j is prime to $c (j=1, \cdots, s)$ and

$$\delta_j \equiv r_j, \mod n, \qquad (j = 1, \cdots, s);$$

and let the c_j be integers >0. Then, if every integer D of the form $c\delta_1^{c_1}\delta_2^{c_2}\cdots\delta_s^{c_s}$ is a determinant of $[a, b, \lambda]$, and if further the infinite sets of solutions appertaining to each pair of unequal determinants of this form are distinct, we shall call the set of all integers of the prescribed form a *discriminant* of $[a, b, \lambda]$, and write this discriminant $\{c, r_1^{c_1}, r_2^{c_2}, \cdots, r_s^{c_s}\}_n$.

Our object is to find discriminants, to exhibit a characteristic $[a, b, \lambda]$ for a given discriminant, to assign the residues of the solutions in each case, and to classify the solutions.

The solutions in certain infinite sets will be classified by separation in finite odd numbers of distinct infinite sets, called *periods*, such that all the solutions in a given period are derivable from a fundamental one, called the *source* of the period. These periods depend upon the following sequences

of integers; in the periods the integers in the sequences are all >0.

Let g, h denote constant integers both different from zero. Then (g, h) is called the *parameter* of the infinity of sequences $w_n (n=0, 1, \cdots)$ of integers defined by

$$w_{n+2} = gw_{n+1} - hw_n$$

and a pair (w_0, w_1) of initial integers. The particular pair of these sequences determined by the pairs of initial values

$$(w_0, w_1) = (0, 1), \quad (w_0, w_1) = (2, g)$$

are the *Lucas sequences* u_n, $v_n (n=0, 1, \cdots)$ respectively, for the parameter (g, h). All the periods in a given infinite set pertain to the same parameter (g, h); different determinants in a given discriminant give periods pertaining to different parameters. As the periods depend upon u_n, v_n, the numerous known properties of Lucas sequences can be applied to read off properties of solutions of $[a, b, \lambda]$. For convenience we add references concerning u_n, v_n.*

2. *Existence Theorems.* Among many others we can state the following three general theorems. These give considerable information regarding the situation described in §1.

There exist discriminants Δ such that, if D is a determinant in Δ, the following statements hold.

THEOREM I. *In addition to determining the characteristic* $[a, b, \lambda]$, *each* D *determines a unique matrix* (T, U) *of integers* $T>0$, $U>0$, *and an odd number* ω *of distinct matrices*

$$(p_j, q_j, \alpha_j, \beta_j, V_j, W_j), \quad (j = 1, \cdots, \omega),$$

in which V_j, W_j *are integers* >0, *such that*

$$\sigma_{jn} \equiv (p_j, q_j, \alpha_j, \beta_j, Uu_n, \mu V_j v_n/2, \mu W_j v_n/2),$$

$$(j = 1, \cdots, \omega; \quad n = 1, 2, \cdots),$$

* Lucas, American Journal of Mathematics, vol. 1, pp. 184, 289; *Théorie des Nombres*, Chapter 18; Bachmann, *Niedere Zahlentheorie*, Kap. 2; Dickson, *History of the Theory of Numbers*, vol. 1, Chapter 17.

are ω distinct infinite sets of solutions of $[a, b, \lambda]$; the u_n, $v_n (n=0, 1, \cdots)$ are the Lucas sequences for the parameter $(2T/\mu, 1)$. The infinite set σ_{jn} $(n=1, 2, \cdots)$ is a period (as described in §1) with the source

$$\sigma_{j1} = (p_j, q_j, \alpha_j, \beta_j, U, TV_j, TW_j),$$

the solution σ_{jn} being obtained from σ_{j1} by multiplying U, TV_j, TW_j by u_n, v_n/v_1, v_n/v_1 respectively. If $\lambda=1$, there is one and only one (q_j, β_j) for each (V_j, W_j).

THEOREM II. *The D as in Theorem* I *also determines a modulus m such that all the solutions $\sigma_{jn} (j=1, \cdots, \omega;\ n=1, 2, \cdots)$ have the same residue modulo m, and this residue remains constant as D ranges over all determinants in Δ.*

THEOREM III. *The matrices (T, U), and hence also the periods described in Theorem* I, *are distinct for unequal determinants D in Δ.*

3. *Specific Theorems for $\lambda=s=c_1=1$.* The relevant parameters to be assigned are c, r, n, which determine Δ, since $s=1$; a, b, which fix $[a, b, \lambda]$ when $\lambda=1$, and m, q_0', which

n, r, c	a, b	m, q_0'	n, r, c	a, b	m, q_0'
(1) 4, 1, 2	1, 1	8, 1	(19) 24, 11, 1	1, 1	12, 7
(2) 8, 1, 1	4, 1	8, 5	(20) 24, 11, 1	1, 2	12, 5
(3) 8, 3, 2	1, 1	8, 5	(21) 24, 13, 1	6, 1	24, 7
(4) 8, 3, 3	4, 1	8, 5	(22) 24, 17, 5	2, 1	24, 11
(5) 8, 3, 4	1, 1	8, 3	(23) 24, 17, 5	18, 1	24, 19
(6) 8, 5, 1	2, 1	8, 3	(24) 24, 19, 1	9, 2	12, 5
(7) 8, 5, 4	1, 1	8, 3	(25) 40, 7, 5	1, 2	20, 13
(8) 12, 5, 4	1, 1	24, 19	(26) 40, 7, 5	1, 2	20, 17
(9) 12, 5, 4	9, 1	24, 11	(27) 40, 11, 1	5, 2	20, 3
(10) 16, 3, 1	8, 1	8, 5	(28) 40, 11, 1	5, 2	20, 7
(11) 16, 7, 1	1, 2	8, 3	(29) 40, 19, 1	5, 2	20, 3
(12) 16, 7, 1	2, 1	8, 5	(30) 40, 19, 1	5, 2	20, 7
(13) 16, 7, 1	4, 1	8, 3	(31) 40, 23, 1	1, 2	20, 3
(14) 16, 7, 2	1, 1	8, 5	(32) 40, 23, 1	1, 2	20, 7
(15) 24, 7, 1	2, 1	24, 5	(33) 40, 23, 5	1, 2	20, 13
(16) 24, 7, 2	1, 1	24, 13	(34) 40, 23, 5	1, 2	20, 17
(17) 24, 7, 2	3, 1	24, 11	(35) 48, 31, 1	18, 1	24, 13
(18) 24, 7, 2	9, 1	24, 5			

give the residue $(*, q_0')_m$, since here p_0' does not occur. We are concerned therefore in this case with §1(3) for assigned values of a, b and primes $q \equiv q_0' \bmod m$; the corresponding values of n, r, c are useful only later in proving that the thirty-five shown in the table give instances of the theorems in §2 applied to §1(3).

From these we see, for example, from the second and third columns for (24), that

$$x^2 - 9y^2 - 2q^\beta z^2 = \mu^2,$$

in which the prime $q \equiv 5 \bmod 12$ has infinities of solutions in the sense of §1, with the properties stated in §2. By the first column the discriminant in this case is the class of all positive primes $\equiv 19 \bmod 24$. The list can be continued indefinitely, and likewise for the next two.

4. *Specific Theorem for* $\lambda = 0$, $s = 1$. As the case $\lambda = 0$ is not fundamentally different from $\lambda = 1$, we give but one example. The discriminant in the following is the class of all positive primes $\equiv 1 \bmod 8$; the primes p, q are both $\equiv 3 \bmod 8$:

$$x^2 - 2p^\alpha y^2 - q^\beta z^2 = \mu^2.$$

5. *Specific Theorems for* $\lambda = 1$, $s = 2$, $c_1 = c_2 = 1$. In each of the following the constant c is unity, so we shall not tabulate it. In the third column the significance of the parenthesis referring to values of q_0' is that one of the two values in the parenthesis is permissible for the accompanying modulus m. Thus we have the entry 40, (11, 19) in (4); hence q in

n, r_1, r_2	a, b	m, q_0'
(1) 8, 1, 3	1, 2	8, 5
(2) 24, 5, 5	12, 1	24, 13
(3) 24, 7, 7	12, 1	24, 13
(4) 40, 3, 7	10, 1	40, (11, 19)
(5) 40, 3, 23	10, 1	40, (11, 19)
(6) 40, 7, 27	10, 1	40, (11, 19)
(7) 40, 23, 27	10, 1	40, (11, 19)
(8) 120, 31, 31	60, 1	120, (61, 109)
(9) 120, 31, 79	60, 1	120, (61, 109)
(10) 120, 79, 79	60, 1	120, (61, 109)

§1(3) is a prime of one of the forms 11 or 19 mod 40 for this example. Since $c_1=c_2=1$ it is necesary in the first column, referring to the discriminant, to give only n, r_1, $r_2(c=1$, as stated).

These illustrate the fact that a given $[a, b, \lambda]$ may have, several discriminants. There are numerous examples in which $s>2$, but we shall omit these and pass to the proofs.

6. *Proofs.* All of the preceding results become obvious on combining two simple remarks. The notation is as in §§1, 2. Take for (T, U) the fundamental solution of the Pellian equation $t^2-Du^2=\mu^2$. Resolve D into the form $ap^\alpha\xi^2+bq^\beta\eta^2$, where ξ, η are integers >0. The rest follows at once. The resolution of D, combined with the successive solutions of the Pellian equation, obtained in the usual manner from (T, U), furnish the periods in §2. There are in each instance ω resolutions of D of the prescribed kind, and hence at least one. That such resolutions exist is known from the ingenious method of Bouniakowsky, or independently from the general arithmetical formulas obtained by paraphrasing identities between elliptic and theta functions. The latter method provides an inexhaustible source of these results and of others of a similar nature relating to forms in more than two variables. Bouniakowsky's method was exploited by Liouville; the specific theorems in §§3–4 can be verified by comparing with Liouville's resolutions of the corresponding D in volumes 3–5 of the second series of his Journal. I believe that proofs for these have not been published; on another occasion I will supply the details.

California Institute of Technology

ON THE EXISTENCE OF LINEAR ALGEBRAS IN BOOLEAN ALGEBRAS*

BY ORRIN FRINK, JR.†

According to the definition to be found in Dickson's *Algebras and their Arithmetics*, pages 9–11, the following properties are characteristic of a linear algebra.

(1) *The elements of the algebra form an abelian group with respect to addition.*

(2) *Multiplication is distributive with respect to addition.*

(3) *The algebra has a finite basis; that is, a finite number of elements can be found such that every element of the algebra can be expressed as a linear combination of these basal units, with coefficients taken from the field over which the algebra is defined.*

With this definition in mind we wish to determine for what pairs of Boolean operations considered as the addition and multiplication operations the elements of a Boolean algebra will constitute a linear algebra. We are limited in our choice of an addition operation by the first property above to operations of the form $axy+a'xy'+a'x'y+ax'y'$, which Bernstein‡ has shown to be the only Boolean abelian group operations. To find suitable multiplication operations $pxy+qxy'+rx'y+sx'y'$, we seek those which are distributive with respect to the above. The condition that the first distributive law hold is found in Schröder's *Algebra der Logik* (vol. 2, p. 503) to be

$$a'(pq+rs)+(ad+a'd'+bc+b'c')(p'q+r's)+d(p'q'+r's')=0,$$

where a, b, c, d, and p, q, r, s are the discriminants of the addition and multiplication operations respectively. Here $a=d$, and $b=c=a'$; hence we get $p'q=r's=a'q=a's=ap'$

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† National Research Fellow.

‡ Transactions of this Society, vol. 26, p. 174.

$=ar'=0$, which can be written $p>a>q$, $r>a>s$. Similarly from the second distributive law we get $p>a>r$, and $q>a>s$. It follows that $q=r=a$, whence it is seen that multiplication must be commutative. Our multiplication operations then must be of the form $pxy+axy'+ax'y+sx'y'$, where $p>a>s$. Another way of writing this which involves only two parameters is $(a+b)xy+axy'+ax'y+ab'x'y'$, where $b=ps'$. It is to be noted that these operations are all associative, the condition being $p's+(p'+s)(qr'+q'r)=0$.

Before trying to satisfy the condition about the finite basis, we will look for *idempotent* elements, that is, elements e not equal to a such that $e^2=e$. Setting $y=x$ in the multiplication operation gives us $(a+b)x+ab'x'$. Equating this to x, we conclude that x is idempotent when and only when it is of the form $(a+b)u+ab'u'$, where u is arbitrary. On the other hand, an element n not equal to a such that $n^2=a$ is *nilpotent*. Equating the above expression for x^2 to a we see that all elements of the form $(a+b')u+abu'$ are nilpotent. If we substitute this expression for a nilpotent element in the multiplication operation we also see that the product of one of these nilpotent elements by any element of the algebra is a. Of course it should be remembered that a corresponds to zero in our algebra.

An important fact is that any element of the algebra can be expressed as the "sum" of an idempotent element and a nilpotent element. For we have $x=(ab'+xb)\oplus(ab+xb')$, where the sign $\oplus$ represents our addition operation as distinguished from logical addition. This is an illustration of the principal theorem on algebras (Dickson, loc. cit., p. 125), which states that any associative linear algebra is the sum of its maximum invariant nilpotent sub-algebra and a semi-simple algebra. The sum is here a direct sum, the cross products being zero.

In our case the field over which the algebra is supposed to be defined is not given to us in advance, as contemplated by the definition of linear algebras. However, an algebra which contains an idempotent element must contain a sub-

algebra which is isomorphic with the field. It can be seen that a and any idempotent e form a field of 2 elements. Bernstein* has shown that the only possible fields in Boolean algebras are of this kind, and in fact if a field contained besides a and e another element f, then since f^2 is idempotent it would have to equal e, since a field cannot contain two idempotents. In that case, however, $e \oplus f$ would be nilpotent, which is likewise impossible in a field. Since we are limited to finite fields, we see that if our algebra is to have a finite basis it must itself be finite. However, any Boolean algebra contains finite Boolean sub-algebras, and hence may contain linear algebras.

We have seen that our entire algebra is the direct sum of a zero algebra and an idempotent algebra. If either b or b' contains only a finite number of elements, then one of these two sub-algebras is finite and may have a finite basis. Suppose the idempotent algebra is finite and contains exactly 2^n elements of the form $ab'+kb$. The product of two such elements $ab'+xb$ and $ab'+yb$ is seen to be $ab'+(xy+axy'+ax'y)b$. We see from this that we may set up a correspondence which is preserved under multiplication, between the elements $ab'+kb$ of our idempotent algebra and the elements kb of a Boolean algebra† whose multiplication operation is $xy+axy'+ax'y$. This Boolean algebra can be represented by the subsets of a set of n elements $(e_1, e_2, \cdots, e_n)$, which combine under logical addition and multiplication. We now pick out the elements kb corresponding to the unit sets (e_1), (e_2), $\cdots$, (e_n) and call them k_1b, k_2b, $\cdots$, k_nb. We then choose as basal units for the idempotent algebra the elements $ab'+k_ib$ $(i=1, 2, \cdots, n)$. The product of any two of these is a if they are distinct. Our field consists of the two elements a and $ab'+a'b$, the latter being the modulus of the algebra. It is seen that these elements correspond to the null set and the universal set

* Transactions of this Society, vol. 28 (1926), p. 654.

† N. Wiener, Transactions of this Society, vol. 18 (1917), p. 65.

in the above representation by means of subsets. To see how an element x of the linear algebra is represented as a linear combination of the basal units, we merely find to what subset of $(e_1, e_2, \cdots, e_n)$ x corresponds. The subscripts of the k's which appear in the linear combination are the same as those of the e's which appear in the subset.

No particular difficulty presents itself in picking a set of basal units for the zero algebra if it is finite. The elements form an abelian group with respect to addition, of order say 2^m, and we need merely select a set of m generators of the group for our basal units. The field, of order 2, must be taken outside the algebra, since the latter does not have a modulus. If the entire algebra is finite, in choosing a set of basal units for it we select the units for the sub-algebras separately, and again we must consider the field to be outside the algebra, since it has no modulus if the nilpotent part exists.

We may sum up our results thus far as follows.

The elements of a Boolean algebra form for the operations $axy+a'xy'+a'x'y+ax'y'$ *and* $(a+b)xy+ax'y+axy'+ab'x'y'$ *considered as the addition and multiplication operations respectively a commutative associative algebra which is the direct sum of a zero algebra and an algebra all of whose elements except the zero element are idempotent. If either of these sub-algebras is finite, it may be represented as a linear algebra over* $GF(2)$, *as may the entire algebra if it is finite.*

These results I consider important for Boolean algebra and the algebra of classes. Let $a=0$ and $b=1$, which gets rid of the zero algebra. Then our addition and multiplication operations become $xy'+x'y$ and xy respectively. In other words we may substitute the operation $xy'+x'y$ for logical addition in Boolean algebra. This operation, which I will for the time being represent with Peano by $x \circ y$, has been treated by many authors, but to Veblen* belongs the credit for introducing it as the addition operation to replace

* Cambridge Colloquium Lectures, p. 9.

logical addition. In the analysis situs applications, where it is called "addition modulo 2," it plays a more important role than logical addition, and its use results in formulas which resemble more those of ordinary algebra.

Two important properties of this operation are $a \oplus a = 0$, and $a' \circ a = 1$ for all values of a. From the first it follows that we may transpose terms from one side of an equation to the other. In virtue of the second property we can avoid entirely the use of the Boolean negation x'. Another important fact is that logical addition and addition modulo 2 are interchangeable when there is no overlapping between terms, as for example in the case of a Boolean expression in the completely expanded form. Thus consider a Boolean function of two variables $f(x, y) = axy + bxy' + cx'y + dx'y'$. Here we could substitute $\circ$ for $+$. If we wish to avoid the use of the negation sign we replace x' by $x \circ 1$ and y' by $y \circ 1$, obtaining $f(x, y) = pxy \circ qx \circ ry \circ d$, where $p = a + b + c + d$, $q = b + d$, and $r = c + d$. This may be considered the normal form for a function of two variables written in terms of addition modulo 2. To pass from an expression of the latter type to one involving logical addition, we may use the well known rule for obtaining the discriminants of a Boolean expression by giving the variables the values 0 and 1. As a final remark it is interesting to note that although we may speak of the logical sum of an infinite number of elements, the sum modulo 2 of an infinite number of elements is in general meaningless.

PRINCETON UNIVERSITY

THE DERIVATION OF ALGEBRAIC INVARIANTS BY TENSOR ALGEBRA*

BY C. M. CRAMLET†

1. *Review of Elements of Tensor Algebra.* The few simple laws of tensor algebra offer a basis for a very natural approach to the theory of algebraic invariants. In the study of algebraic invariants a ground form is chosen as, for example, the bilinear n-ary form $a_{\rho\sigma}u^\rho v^\sigma$‡ and the quantities u^r and v^s are transformed linearly by cogredient transformations such as

$$\bar{u}^r = u^\rho q_\rho^{\ r}. \tag{1}$$

If the equivalent transformed form be now written $\bar{a}_{\rho\sigma}\bar{u}^\rho\bar{v}^\sigma$, the transformation equations for the a's are

$$\bar{a}_{rs} = a_{\rho\sigma}p_r^{\ \rho}p_s^{\ \sigma}, \tag{2}$$

where the p's satisfy the equations

$$p_s^{\ r}q_\rho^{\ r} = p_\rho^{\ r}q_s^{\ \rho} = \delta_s^r, \qquad (= 0, r \neq s\,;\ = 1, r = s), \tag{3}$$

or what is equivalent, $q_s^{\ r}$ is equal to the cofactor of $p_s^{\ r}$ in the determinant $|p_s^r|$, divided by this determinant. In tensor algebra the two equations of transformation (1) and (2) illustrate two types of tensors, that is, sets of ordered functions which are transformed in this linear manner. The set of quantities u^ρ is called a *contravariant tensor of rank* 1 since the transformed quantities are each expressed linearly in the q's. The set of quantities a_{rs} with two lower indices

* Presented to the Society, February 26, 1927, except §2.

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‡ Repeated Greek letters are summed from 1 to n. Such a form is ordinarily written $a_{\rho\sigma}x^\rho y^\sigma$. In the theory of surfaces, with the constant values at a point, of a given tensor such as a_{rs} may be associated arbitrary vectors, in this case two, u^r and v^s. Thus when a general transformation of coordinates is made the vectors are transformed linearly, so the theory of algebraic invariants is applicable at the point.

illustrate a *covariant tensor of rank* 2. These terms *covariant* and *contravariant* are used differently in the classical theory of invariants. Here the terminology of tensor algebra will be used.

The three laws of tensor algebra which follow directly from the defining transformation equations are as follows.

(a) *The sum of two tensors of the same kind is a tensor of that kind.*

(b) *The (outer) product of two tensors is a tensor whose rank is the sum of the ranks of these tensors.* For example, $a_{rs}v^t$ is a mixed tensor covariant of rank 2 and contravariant of rank 1. In the notation $T_{rs}^{\ t}$, the transformation law is

$$\overline{T}_{rs}^{\ t} = T_{\rho\sigma}^{\ \tau} q_\tau^t p_r^\rho p_s^\sigma.$$

(c) *If a covariant and contravariant index of a mixed tensor such as $T_{rs}^{\ t}$ are set equal and summed, as the notation $T_{r\alpha}^{\ \alpha}$ indicates, a tensor is obtained which has its contravariant and covariant rank each decreased by one.* The proof consists in setting $t=s=\alpha$ in the equation of (b) above, and making use of (3), from which we find

$$\overline{T}_{r\alpha}^{\ \alpha} = T_{\rho\sigma}^{\ \ \tau} q_\tau^\alpha p_r^\rho p_\alpha^\sigma = T_{\rho\sigma}^{\ \ \tau} p_r^\rho \delta_\tau^\sigma = T_{\rho\alpha}^{\ \ \alpha} p_r^\rho.$$

This is the transformation equation defining a covariant vector. The expression $a_{\rho\sigma}u^\rho v^\sigma$ having no unsummed indices is called a scalar. The process is called contraction or inner multiplication and the contracted or summed indices are said to cancel.

The symbol δ_s^r which occurs in (3) may be made a tensor by assigning the law of transformation to it that its indices suggest. Then $\overline{\delta}_s^r = \delta_\sigma^\rho p_s^\sigma q_\rho^r = p_s^\rho q_\rho^r = \delta_s^r$, by (3) and the new components are exactly equal to the old components. In general therefore the tensor defined by the equation

$$(4) \quad \gamma_{s_1 \cdots s_k}^{r_1 \cdots r_k} = B^1 \delta_{s_1}^{r_1} \delta_{s_2}^{r_2} \cdots \delta_{s_k}^{r_k} + B^2 \delta_{s_1}^{r_2} \delta_{s_2}^{r_1} \cdots \delta_{s_k}^{r_k} + \cdots,$$

in which every permutation of the set r appears and the B's are arbitrary scalars, possesses this property. It has been

proved* moreover that there are no other invariant tensors. The tensor $\delta^{r_1\cdots r_k}_{s_1\cdots s_k}$, obtained from (4) by assigning B the values ± 1 according as the permutation on r of the δ's of which B is the coefficient is even or odd, since it transforms by invariance will (as does $\gamma^{r_1\cdots r_k}_{s_1\cdots s_k}$) satisfy the equation

$$\delta^{r_1\cdots r_k}_{s_1\cdots s_k} = \delta^{\rho_1\cdots\rho_k}_{\sigma_1\cdots\sigma_k} q^{\sigma_1}_{s_1}\cdots q^{\sigma_k}_{s_k} p^{r_1}_{\rho_1}\cdots p^{r_k}_{\rho_k}. \tag{5}$$

Setting $k=n$ and $r_1\cdots r_n = 1\cdots n$, this becomes

$$\delta^{1\cdots n}_{s_1\cdots s_n} = \delta^{1\cdots n}_{\sigma_1\cdots\sigma_k} p^{\sigma_1}_{s_1}\cdots p^{\sigma_n}_{s_n}\cdot \left| q_s{}^r \right|.$$

This is usually written as

$$\epsilon_{s_1\cdots s_n} = \epsilon_{\sigma_1\cdots\sigma_n} p^{\sigma_1}_{s_1}\cdots p^{\sigma_n}_{s_n}\cdot \left| q_s{}^r \right|, \tag{6}$$

and is called a covariant tensor† density of weight -1. This equation represents its transformation equation and shows these important facts: (1) it is invariant under a transformation of coordinates; (2) the indices are of tensor character, that is, indicate a certain law of transformation; (3) a factor Δ^{-1} must be introduced after a transformation of coordinates. Similarly from (5), setting the indices s equal to $1\cdots n$, we have

$$\epsilon^{r_1\cdots r_n} = \epsilon^{\rho_1\cdots\rho_n} q^{r_1}_{\rho_1}\cdots q^{r_n}_{\rho_n}\cdot\Delta, \tag{7}$$

and $\epsilon^{r_1\cdots r_n}$ is called a contravariant tensor density of weight 1.

2. *Proof of the Fundamental Theorems.*‡ It will now be

* P. Franklin, Philosophical Magazine, vol. 45, p. 998; C. M. Cramlet, this Bulletin, vol. 32 (1926), p. 212; T. Y. Thomas, Annals of Mathematics, vol. 27, No. 4 (1926), p. 548; C. M. Cramlet, Tôhoku Mathematical Journal, vol. 28, Nos. 3, 4 (1927), p. 242.

† See Veblen, *Invariants of Quadratic Differential Forms*, Cambridge Tract No. 24, 1927, Chapter I and Chapter II, §14.

‡ The problem of §2 was discussed in a series of conferences with Professor O. Veblen and Mr. W. Flexner. Thanks are due to Professor Veblen for constructive criticism which was helpful for a clear formulation of the problem. Mr. Flexner has worked out a proof of this problem by the use of the Cayley operator. The novel feature of his proof is the fact that the symbolic method is not used.

shown that the determination of all algebraic invariants reduces to an exercise of the three laws of tensor algebra.

Let us suppose that we have a system of forms determined by the coefficients (or covariant tensors) $a_{r_1 \dots r_h}$, $b_{r_{h+1} \dots r_i}$, $\cdots$, $c_{\dots r_k}$. Assuming the theorems, which depend upon elementary considerations on the arbitrariness of these forms, that only functions homogeneous in each of the tensors separately need be considered, we know that a typical term of an algebraic invariant is

$$A_{r_1 \cdots r_k} \equiv a_{r_1 \cdots r_h} b_{\cdots r_i} \cdots c_{\cdots r_k},$$

and the invariant is representable as

$$I = K^{\rho_1 \cdots \rho_k} A_{\rho_1 \cdots \rho_k} \tag{8}$$

the K's being undetermined constants. An algebraic invariant must by definition be transformed according to the equation $\bar{I} = I \cdot F$, where $\bar{I}$ is exactly the same function of the coefficients of the transformed forms as is I of the original forms. Assuming further the well known theorem, which is a consequence of the assumption that I is rational, that F is an integral power of the modulus of the transformation, say Δ^w where $\Delta = |p_j^i|$, we may write this equation in the form

$$K^{\rho_1 \cdots \rho_k} \bar{A}_{\rho_1 \cdots \rho_k} = K^{\rho_1 \cdots \rho_k} A_{\rho_1 \cdots \rho_k} \Delta^w. \tag{9}$$

On the other hand, since A is a tensor, we have

$$K^{\rho_1 \cdots \rho_k} \bar{A}_{\rho_1 \cdots \rho_k} = K^{\rho_1 \cdots \rho_k} A_{\sigma_1 \cdots \sigma_1} p_{\rho_1}^{\sigma_1} \cdots p_{\rho_k}^{\sigma_k}. \tag{10}$$

The right members of these equations are identically equal for arbitrary p's; hence $k = wn$. Multiplying (6) by Δ and (9) by w ϵ's, we find, by direct substitution of this new (6) into this new (9),

$$\begin{aligned} &\epsilon_{r_1 \cdots r_n} \cdots \epsilon_{\cdots r_k} K^{\rho_1 \cdots \rho_k} \bar{A}_{\rho_1 \cdots \rho_k} \\ &\qquad = K^{\sigma_1 \cdots \sigma_k} A_{\sigma_1 \cdots \sigma_k} \epsilon_{\rho_1 \cdots \rho_n} \cdots \epsilon_{\cdots \rho_k} p_{r_1}^{\rho_1} \cdots p_{r_k}^{\rho_k}. \end{aligned} \tag{11}$$

Substituting in this

$$A_{\sigma_1\cdots\sigma_k} = \overline{A}_{\tau_1\cdots\tau_k} q^{\tau_1}_{\sigma_1} \cdots q^{\tau_k}_{\sigma_k},$$

we have

$$(12)\quad {}_{\tau_1}\overline{A}\ldots_{\tau_k}\left[\epsilon_{\tau_1\cdots\tau_n}\cdots\epsilon\ldots_{\tau_k}K^{\tau_1\cdots\tau_k} - \epsilon_{\rho_1\cdots\rho_k}\cdots\epsilon\ldots_{\rho_k}K^{\sigma_1\cdots\sigma_k}q^{\tau_1}_{\sigma_1}\cdots q^{\tau_k}_{\sigma_k}p^{\rho_1}_{\tau_1}\cdots p^{\rho_k}_{\tau_k}\right]=0.$$

If $\overline{A}$ is an arbitrary tensor its coefficient must vanish. But A may be a product of tensors, in which case the terms of the product are interchangeable, or, one of the tensors may determine a p-ic in which case its indices are interchangeable. There is no loss of generality in the invariant I defined by equation (8) if the K's are assumed to have the same symmetry as the A's. This amounts to replacing the constants associated with equal terms by their average. We may now set coefficients of $\bar{A}$ equal to zero as in the case where $\bar{A}$ is wholly arbitrary. The vanishing of these coefficients expresses the conditions that $\epsilon_{\tau_1\ldots\tau_n}\cdots\epsilon_{\ldots\tau_k}K^{\tau_1\cdots\tau_k}$ be an invariant tensor of the rank indicated by the indices, that is a tensor having identical components in all systems of coordinates. Using the theorem of §1 embodied in (4), we find

$$\epsilon_{\tau_1\cdots\tau_n}\cdots\epsilon\ldots_{\tau_k}K^{\rho_1\cdots\rho_k} = B^1\delta^{\rho_1}_{\tau_1}\delta^{\rho_2}_{\tau_1}\cdots\delta^{\rho_k}_{\tau_k} + B^2\delta^{\rho_2}_{\tau_1}\delta^{\rho_1}_{\tau_2}\cdots\delta^{\rho_k}_{\tau_k} + \cdots.$$

Multiplying through by $\epsilon^{\tau_1\cdots\tau_n}\cdots\epsilon^{\cdots\tau_k}$, τ summed, we have

$$(13)\ (n!)^w K^{\rho_1\cdots\rho_k} = B^1\epsilon^{\rho_1\rho_2\cdots\rho_n}\cdots\epsilon^{\cdots\rho_k} + B^2\epsilon^{\rho_1\rho_2\cdots\rho_n}\cdots\epsilon^{\cdots\rho_k} + \cdots.$$

Since $k=wn$, each term contains w ϵ's and the ρ's appear in all possible permutations. We have thus found that for I of (8) to be an invariant it is necessary that $K^{\rho_1\cdots\rho_k}$ be of the form (13). It is obviously sufficient, for when each term of (13) with the B's arbitrary is multiplied by inner multiplication with $A_{\rho_1\ldots\rho_k}$ the tensor indices cancel by inner multiplication according to tensor law (c), §1, and, since $\epsilon^{\rho_1\cdots\rho_n}$ is a tensor density of weight one, each term gives rise to a scalar density of weight w. Those solutions that do not satisfy the symmetry conditions imposed on K,

that is, those which have two antisymmetric indices on an ϵ summed with two symmetric indices on an A, will vanish identically. We thus have the following theorem.

THEOREM 1. *An algebraic invariant of weight w is formed from a given system of covariant tensors by forming by outer multiplication a tensor A of rank wn and multiplying this by inner multiplication by w ϵ's, associating the indices of the ϵ's in any manner whatever with the indices of A. Moreover these constitute all algebraic invariants that can be formed from the given covariant tensors.*

The extension to the following general theorem is immediate.

THEOREM 2. *From a given system of covariant and contravariant tensors, a tensor $A_{r_1\cdots r_k}^{s_1\cdots s_l}$ may be formed by outer and inner multiplication. In this tensor, k and l are multiples of n, say, $k=w_1n$, $l=w_2n$ and $w=(k-l)/n$, all letters representing integers. All algebraic invariants may be formed by multiplying by inner multiplication w_1 of the contravariant ϵ's, $\epsilon^{r_1\cdots r_n}$ and w_2 covariant ϵ's, $\epsilon_{r_1\cdots r_n}$.* A few illustrations of the methods here described will be given.

3. *Algebraic Invariants of a System of Linear Forms.* It will require n tensors such as a_r to build up an nth rank tensor by outer multiplication. These n tensors can be represented by $a_{r|s}$. The bar is used to indicate that the preceding r does not designate a tensor character. The only possible algebraic invariant is $\epsilon^{\rho_1\cdots\rho_n}a_{r_1|\rho_1}\cdots a_{r_n|\rho_n}$.* This is merely the determinant of the system of forms. For a set of vectors $u_{r|}$* the algebraic invariant of weight -1 is

$$\epsilon_{\sigma_1\cdots\sigma_n}u_{r_1|}{}^{\sigma_1}\cdots u_{r_n|}{}^{\sigma_n}.$$

* The non-tensor range is alternating here. The number of alternating ranges associated with n elements is always even. See my paper, *Applications of the determinant and permanent tensors to determinants of general class and allied tensor functions*, American Journal, vol. 49, No. 1 (January, 1927), p. 87.

4. *Symmetry Tensors.* When two or more indices of an element are summed with indices of the same δ or ϵ there results an algebraic invariant whose vanishing expresses a symmetry condition on the ground form. For example, if the vectors in the bilinear form $a_{\rho\sigma}u^{\rho}v^{\sigma}$ have but two components ($n=2$, the binary bilinear form), we may form the algebraic invariant $\delta^{\rho\sigma}_{rs}a_{\rho\sigma}$. The vanishing of this indicates that $a_{rs}=a_{sr}$ in all coordinate systems. The property remains true for any n but then we have a tensor that is not an algebraic invariant. The property of being skew symmetric is a property that carries over to any new form yet the condition is not expressible as an algebraic invariant but only as a tensor. This condition is expressed by the tensor equation

$$\pi^{\rho\sigma}_{rs}\,a_{\rho\sigma}=0,\ (\pi^{\rho\sigma}_{rs}=\delta^{\rho}_{r}\delta^{\sigma}_{s}+\delta^{\sigma}_{r}\delta^{\rho}_{s}),\quad \text{or}\quad a_{rs}=-a_{sr}.$$

In a new coordinate system this becomes $\pi^{\rho\sigma}_{rs}\bar{a}_{\rho\sigma}=0$, from which we find $\bar{a}_{rs}=-\bar{a}_{sr}$. The method of obtaining other types of symmetry conditions for tensors with the use of the δ or γ tensors is apparent.*

5. *Bilinear Forms.* With the two-index tensor a_{rs} the following invariant of weight one may be formed if n is even (equal to $2i$, say):

$$\epsilon^{\rho_1\sigma_1\cdots\rho_i\sigma_i}a_{\rho_1\sigma_1}\cdots a_{\rho_i\sigma_i}. \tag{14}$$

If we seek the invariants of weight two, the degree must equal n, for example

$$\epsilon^{\rho_1\cdots\rho_n}\epsilon^{\sigma_1\cdots\sigma_n}a_{\rho_1\sigma_1}\cdots a_{\rho_n\sigma_n}. \tag{15}$$

The set of indices either ρ or σ on the elements is called a range. If the tensor is not symmetric the indices of an ϵ need not be summed with a range. There will then be many invariants like (15). If, however, they are summed with a range, the second ϵ merely multiplies the invariant by $n!$.

6. *Systems of Bilinear Forms.* Explicit expressions for a

* Weyl, *Space, Time, Matter*, 4th edition, pp. 57, 58.

system of bilinear forms are readily written. Represent the system by

$$\phi_{q1} = a_{q|\rho\sigma}u^{\rho}v^{\sigma}, \qquad (q = 1,2, \cdots).$$

The following is a set of invariants of the forms:

$$\epsilon^{\rho_1 \cdots \rho_n}\epsilon^{\sigma_1 \cdots \sigma_n}a_{q_1|\rho_1\sigma_1} \cdots a_{q_n|\rho_n\sigma_n}. \tag{16}$$

The q's may be distinct or repeated (see reference in §3) and any of the elements may be outer products of elements of linear forms.

7. *Trilinear Forms.* With a given tensor a_{rst} it is only necessary to construct inner tensor products with the tensors $\epsilon^{\rho_1 \cdots \rho_n}$ so that all indices on the elements are summed. All such are algebraic invariants and all algebraic invariants are thus acquired. To illustrate in detail a simple case, choose $n=2$ and require umbral letters on an ϵ to be associated with indices of a range. Such invariants will also be invariants of the cubic, for which a_{rst} is symmetrical. Thus consider the invariant of degree 4 and weight 6 of the binary cubic

$$\begin{aligned} I &= \epsilon^{\alpha_1\alpha_2}\epsilon^{\beta_1\beta_2}\epsilon^{\alpha_3\alpha_4}\epsilon^{\beta_3\beta_4}\epsilon^{\gamma_1\gamma_4}\epsilon^{\gamma_2\gamma_3}a_{\alpha_1\beta_1\gamma_1}a_{\alpha_2\beta_2\gamma_1}a_{\alpha_3\beta_3\gamma_3}a_{\alpha_4\beta_4\gamma_4} \\ &= \epsilon^{\alpha_1\alpha_2}\epsilon^{\beta_1\beta_2}a_{\alpha_1\beta_1\gamma_1}a_{\alpha_2\beta_2\gamma_2}\cdot\epsilon^{\alpha_3\alpha_4}\epsilon^{\beta_3\beta_4}a_{\alpha_3\beta_3\gamma_3}a_{\alpha_4\beta_4\gamma_4}\cdot\epsilon^{\gamma_1\gamma_4}\epsilon^{\gamma_2\gamma_3}. \end{aligned} \tag{17}$$

The product of the first four quantities in this equation is a cubic determinant so the indices γ_1 and γ_3 are a permanent range. We may represent this product by $A_{\gamma_1\gamma_3}$. The whole theory of algebraic invariants might be translated into theorems regarding inner products of higher determinants. Further details of evaluating this invariant will be carried out:

$$I = \epsilon^{\gamma_1\gamma_4}A_{\gamma_1\gamma_2}A_{\gamma_4\gamma_3}\epsilon^{\gamma_2\gamma_3}.$$

The first three terms of this are again a determinant, hence the range $\gamma_2\gamma_3$ is alternating and yields when multiplied innerly with $\epsilon^{\gamma_2\gamma_3}$, $I = 2!\epsilon^{\gamma_1\gamma_4}A_{\gamma_1\gamma_3}A_{\gamma_4\gamma_2}$. This invariant is determined by the particular manner in which the indices

of an ϵ are associated with the indices of the elements; thus schematically we may write 17 as

$$I = (13)(13)(24)(24)(14)(23) = (13)^2(24)^2(14)(23).$$

This expression is 2! times the invariant represented in the same manner by the symbolic method.

8. *Conclusion.* We have here laid down a basis for a development of the theory of algebraic invariants which is applicable to tensors of any type. Transition can be made to the symbolic method which now becomes established without the use of ideal vectors. The computation seems particularly more simple than for the symbolic method.

The ϵ symbols have been used throughout and the number of indices on an ϵ was always the same as the number of components of a vector in n-space or as the number of variables used to represent a point in $(n-1)$-space in homogeneous coordinates. If in any of the formulas we replace an $\epsilon^{r_1 \cdots r_n}$ or $\epsilon_{s_1 \cdots s_n}$ by a $\delta^{r_1 \cdots r_k}_{s_1 \cdots s_k}$ and allow k to be less than n, each algebraic invariant becomes an alternating tensor which has 1 distinct component for $k=n$ and more components for $k<n$. This observation shows rather clearly the position algebraic invariants occupy in tensor algebra and again among alternating tensors. The reason that algebraic invariants have been of greater interest lies in the fact that the vanishing of an algebraic invariant gives but one condition while the vanishing of any other tensor is a much stronger condition since it gives rise to as many conditions as the tensor has distinct components.*

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*F. L. Hitchcock has written a number of papers bearing upon the subject of this paper, in particular see *A new method in the theory of quantics*, Journal of Mathematics and Physics, vol. 4 (1925), p. 238. If his view point is adopted in this paper, the results remain true if the indices associated with each ϵ and with the indices of A that are multiplied innerly with this ϵ are cogredient, that is, we know that all invariants are obtained by these methods. Further we know that all such upon specialization by making all indices cogredient become algebraic invariants. Then, however, the question would still remain open as to whether these included all algebraic invariants.

TRANSVERSE SEISMIC WAVES ON THE SURFACE OF A SEMI-INFINITE SOLID COMPOSED OF HETEROGENEOUS MATERIAL*

BY H. BATEMAN

In his Adams' Prize Essay of 1910, A. E. H. Love† discussed the propagation of transverse seismic waves in a homogeneous superficial layer covering a homogeneous solid differing from the surface layer in rigidity and density. The analysis has been extended by K. Aichi‡ and E. Meissner§ to the case of a heterogeneous material stratified in horizontal layers. Aichi adopts an exponential law for the variation of secondary wave velocity‖ with depth and obtains a solution involving Bessel functions. Meissner considers various laws of variation of density and rigidity, and in one case he also obtains a solution involving Bessel functions. In this paper, a solution is given for a heterogeneous material of such a kind that the functions occurring are all of an elementary nature.

Using the same notations as Aichi, but taking the positive direction of the axis of z upwards, the axis of x being in the direction of propagation, we assume that the transverse displacement η is given by the equation

$$\eta = Y(z) \cos (pt + fx), \tag{1}$$

and that the density ρ and rigidity μ of the material depend only on the coordinate z.

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* Presented to the Society, San Francisco Section, October 29, 1927.

† *Some Problems of Geodynamics*, Cambridge University Press, 1911, p. 160.

‡ Proceedings of the Physico-Mathematical Society of Japan, vol. 4 (1922), pp. 137–142.

§ Proceedings of the Second International Congress for Applied Mathematics, Zürich, 1926.

‖ This is the quantity c defined by equation (5).

The components of stress across an area perpendicular to the axis of y are $\mu\partial\eta/\partial x$, 0, $\mu\partial\eta/\partial z$, respectively, and so the equation of motion

$$\rho\frac{\partial^2\eta}{\partial t^2}=\frac{\partial}{\partial x}\left(\mu\frac{\partial\eta}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu\frac{\partial\eta}{\partial z}\right)$$

gives Meissner's equation

$$\frac{d}{dz}\left(\mu\frac{dY}{dz}\right)+(\rho p^2-\mu f^2)Y=0. \tag{2}$$

When the rigidity μ is constant, this equation reduces to the equation

$$\frac{d^2Y}{dz^2}+\left(\frac{\rho p^2}{\mu}-f^2\right)Y=0, \tag{3}$$

used by Aichi.

In addition to the differential equation, Y must satisfy the boundary conditions $Y=0$ when $z=-\infty$, $dY/dz=0$ when $z=a$. The first of these conditions expresses simply that there is no deep penetration of the waves while the second condition expresses that there is no tangential stress at the free surface $z=a$.

Taking first the case of a constant value of μ, we write

$$\frac{\rho p^2}{\mu}=2n^2\operatorname{sech}^2 nz+k^2,\qquad f^2=n^2\operatorname{ctnh}^2\omega+k^2, \tag{4}$$

where k^2, ω, and n are positive constants when p is given. Our law of variation of ρ thus apparently depends on the frequency of the waves, and is definite only when this frequency is assigned. We shall therefore regard this particular p as a constant, and we shall use a different symbol P when the frequency is varied.

It is easily seen that the differential equation and the first boundary condition are satisfied by the equation

$$Y=e^{nz\operatorname{ctnh}\omega}\operatorname{sech} nz\cosh(nz-\omega).$$

The second boundary condition is satisfied if

$$\sinh^2\omega\tanh^2 na-\sinh\omega\cosh\omega\tanh na+1=0.$$

Regarding this as an equation for tanh na when ω is given, we find that if $\cosh \omega > 2$ there are two real values of tanh na, both of which are positive. This means that at first ρ increases with the depth. Now c, the velocity of propagation of "S" waves, is given by the formula

$$c^2 = \frac{\mu}{\rho}. \tag{5}$$

Hence at first c decreases downwards reaching a minimum value at the plane $z=0$. Below this plane c increases with the depth and gradually approaches a limiting value which is greater than the value at the surface. The existence of a solution in the present case is particularly interesting because in Love's case a solution exists only when c is greater for the substratum than for the upper layer. Let b denote the value of c at the surface and v the velocity of the transverse waves represented by (1); then we have

$$v = \frac{p}{f}, \quad \frac{p^2}{b^2} = 2n^2 \operatorname{sech}^2 na + k^2, \quad f^2 = n^2 \operatorname{ctnh}^2 \omega + k^2 ;$$

but

$$2 \operatorname{sech}^2 na = \operatorname{ctnh}^2 \omega \mp \operatorname{ctnh} \omega (\operatorname{ctnh}^2 \omega - 4 \operatorname{csch}^2 \omega)^{1/2} ;$$

therefore

$$\frac{p^2}{b^2} - f^2 = \mp n^2 \operatorname{ctnh} \omega (\operatorname{ctnh}^2 \omega - 4 \operatorname{csch}^2 \omega)^{1/2}.$$

This equation tells us that v is greater or less than b according as the lower or upper sign is given to the square root. When the upper sign is taken, the amplitude Y begins to increase downwards until it attains a maximum value at a level z given by the equation

$$2 \sinh \omega \tanh nz = \cosh \omega - (\cosh^2 \omega - 4)^{1/2} ;$$

beyond this level the amplitude decreases steadily to zero. When the lower sign is taken the amplitude decreases steadily to zero from the surface downwards. To discuss the effect

of a change of frequency, we consider the propagation of the disturbance

$$\eta = Y(z)\cos(Pt + Fx),$$

the law of variation of density with depth being the same as before. Writing

$$2P^2 = m(m+1)p^2,\quad 2F^2 = m(m+1)k^2 + 2\sigma^2 n^2,$$

where m and σ depend on P but p is constant as before, we find that Y must now satisfy the differential equation

$$\frac{d^2Y}{dz^2} + n^2 Y[m(m+1)\operatorname{sech}^2 nz - \sigma^2] = 0. \tag{6}$$

A particular solution is given by the hypergeometric function

$$Y = e^{n\sigma z}F(-m,\ m+1\ ;\ 1-\sigma\ ;\ \xi), \tag{7}$$

where

$$2\xi = 1 - \tanh nz.$$

When m is an integer, this function is a polynomial in ξ except for the exponential factor. If σ is positive, Y will satisfy the boundary condition at $z=-\infty$. The boundary condition at $z=a$ gives a relation between σ and the quantity $\tanh na$, which will be denoted by the symbol τ. In the cases $m=1, 2, 3$, the relations are

$$\sigma^2 - \sigma\tau + \tau^2 = 1,$$

$$\sigma^3 - 3\sigma^2\tau + 6\sigma\tau^2 - 6\tau^3 = 4\sigma - 6\tau,$$

$$\sigma^4 - 6\sigma^3\tau + 21\sigma^2\tau^2 - 45\sigma\tau^3 + 45\tau^4 = 10\sigma^2 - 39\sigma\tau + 54\tau^2 - 9,$$

respectively. Portions of the curves represented by these equations are drawn roughly in Fig. 1, σ and τ being rectangular coordinates.

FIG. 1

For large values of m there are several possible values of σ corresponding to a given value of τ; thus when $m=3$ and $\tau=1$, we can have $\sigma=0$, 1, 2, or 3.

For all integral values of m, unity is a possible value of σ corresponding to $\tau=1$, and when τ is very nearly equal to one, there is a value of σ which is very nearly equal to one.

An approximate expression for this value of σ may be derived from equation (7) by differentiating with respect to z and retaining only first powers of $1-\tau$ after z has been put equal to a. The resulting equation,

$$0 = \sigma\left[1 - \frac{m(m+1)}{2(1-\sigma)}(1-\tau)\right] + (1-\tau^2)\left[\frac{m(m+1)}{2(1-\sigma)}\right],$$

gives approximately

$$\sigma = 1 + \frac{m(m+1)}{2}(1-\tau).$$

Using this value of σ, we have approximately

$$\begin{aligned} 2F^2 &= 2n^2 + m(m+1)\left[k^2 + n^2(1-\tau)\right] \\ &= 2n^2 + 2\frac{P^2}{p^2}\left[k^2 + n^2(1-\tau)\right]. \end{aligned}$$

Now the wave-velocity v and the group-velocity u are given by the formulas,*

$$\frac{1}{v} = \frac{F}{P} = (pP)^{-1}\left[n^2p^2 + P^2k^2 + P^2n^2(1-\tau)\right]^{1/2},$$

$$\frac{1}{u} = \frac{dF}{dP} = Pp^{-1}\left[k^2 + n^2(1-\tau)\right] \cdot \left[n^2p^2 + P^2k^2 + P^2n^2(1-\tau)\right]^{-1/2},$$

$$\frac{u}{v} = 1 + \frac{n^2p^2}{P^2k^2 + P^2n^2(1-\tau)}.$$

Since $1-\tau$ is positive the group-velocity is greater than the

* H. A. Lorentz, *Problems of Modern Physics*, p. 17.

wave-velocity. This result is fundamentally different* from that obtained in Love's case of two adjacent homogeneous layers, and also from that obtained by Meissner for the case of a heterogeneous medium in which the density and rigidity vary continuously and monotonically. Some doubt may be felt with regard to the validity of our approximation and the differentiation with respect to P and it is hoped that the result will be checked by some independent method.

Turning now to the case in which the rigidity varies with the depth, we put

$$Z = \mu^{1/2} Y ;$$

the differential equation satisfied by Z is then

$$\frac{d^2Z}{dz^2} + Z\left[\frac{\rho P^2}{\mu} - F^2 + \frac{1}{2\mu}\frac{d^2\mu}{dz^2} - \frac{1}{4\mu^2}\left(\frac{d\mu}{dz}\right)^2\right].$$

This differential equation may be reduced to the type already considered in the following cases:

(i) $\mu = \nu e^{-2kz}$,

$$\frac{\rho p^2}{\mu} = 2\omega^2 \operatorname{sech}^2 nz + k^2,$$

$$F^2 = \frac{P^2 k^2}{p^2} + \sigma^2 n^2 + h^2,$$

(ii) $\mu = \nu \cosh nz$,

$$\frac{\rho p^2}{\mu} = 2\omega^2 \operatorname{sech}^2 nz + k^2,$$

$$F^2 = \frac{P^2 k^2}{p^2} + \frac{n^2}{4} + \sigma^2 n^2,$$

where ω is an arbitrary constant.

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* The group velocity is generally less than the wave-velocity. This is shown very clearly in Meissner's diagrams but Meissner's definition of the group-velocity is different from ours.

ON CONTINUOUS CURVES IN n DIMENSIONS*

BY G. T. WHYBURN AND W. L. AYRES†

If M_1 and M_2 are subsets of a connected point set M, the subset K of M is said to *separate* M_1 and M_2 in M if $M-K$ is the sum of two mutually separated sets containing M_1 and M_2 respectively. R. L. Moore‡ has shown that in order that a plane continuum M be a continuous curve§ it is necessary and sufficient that for every two distinct points A and B of M there should exist a subset of M which consists of a finite number of continua and which separates A and B in M. Consider the following example: Let S_i $(i=1, 2)$ be the set of all points (x, y, z) in three dimensions such that $x=(-1)^i$, $-1\leqq y\leqq 1$, $0\leqq z\leqq 1$. Let R_0 be the set of all points (x, y, z) such that $-1\leqq x\leqq 1$, $-1\leqq y\leqq 1$, $z=0$. For each integer $n>0$, let R_n be the set of all points (x, y, z) such that $-1\leqq x\leqq 1$, $-1\leqq y\leqq 1$, $z=1/n$. Let

$$M = S_1 + S_2 + \sum_{n=0}^{\infty} R_n.$$

It is easy to see that every two points of M may be separated by a single subcontinuum of M and yet M is not a continuous curve. Hence the condition given by Moore is not sufficient in order that a continuum in n dimensions $(n>2)$ be a continuous curve. In this paper we give two modifications (Theorems 2 and 4) of Moore's theorem which hold in n dimensions.

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† National Research Fellow in Mathematics.

‡ *A characterization of a continuous curve*, Fundamenta Mathematicae, vol. 7 (1925), pp. 302–307.

§ We shall use the term *continuous curve* in the sense of a point set which is closed, connected and connected im kleinen. See R. L. Moore, *Concerning simple continuous curves*, Transactions of this Society, vol. 21 (1920), p. 347.

THEOREM 1.* *If M is a continuous curve in euclidean space of n dimensions, K is a bounded subcontinuum of M and ϵ is any positive number, then there exists a set L such that* (1) *$K+L$ is a continuous curve which is a subset of M,* (2) *every point of L is within a distance ϵ of some point of K,* (3) *L consists of a countable set of arcs of M, not more than a finite number of which are of diameter greater than any given positive number,* (4) *$L+K$ is non-dense at every point except those points at which K fails to be non-dense.*

PROOF. The set M is uniformly connected im kleinen over the set K.† Let $\delta_1, \delta_2, \delta_3, \cdots$ be a sequence of positive numbers such that every two points of K whose distance from one another is less than δ_m can be joined by an arc of M whose diameter is less than $\epsilon/2m$. For each point p of K and each positive integer n, let C_{np} and C'_{np} be hyperspheres with center p and radii ϵ/n and $\epsilon/(2n)$ respectively.‡ By the Borel theorem, for each value of n there is a finite subset of the set $[C'_{np}]$,

$$C'_{np_{n1}}, C'_{np_{n2}}, C'_{np_{n3}}, \cdots, C'_{np_{nn'}},$$

such that every point of K is in the interior of one of the sets $C'_{np_{ni}}$ for $1 \leqq i \leqq n'$. Since M is a continuous curve there are but a finite number,

$$M_{ni1}, M_{ni2}, M_{ni3}, \cdots, M_{nim_n},$$

of the components§ of $M \cdot I(C_{np_{ni}})$ that contain points in the interior of $C'_{np_{ni}}$. For each n, i and j, let $[K_{nij}]$ be the set of

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* This theorem contains as a special case a theorem due to H. M. Gehman, *Concerning the subsets of a plane continuous curve*, Annals of Mathematics, vol. 27 (1925), pp. 29–46, Theorem 3.

† S. Mazurkiewicz, *Sur les lignes de Jordan*, Fundamenta Mathematicae, vol. 1 (1920), p. 173.

‡ If p is a point and r a positive number, the *hypersphere* with center p and radius r is the set of all points of the space whose distance from the point p is r. If S is a hypersphere, $I(S)$ denotes the interior of S.

§ A connected subset of a point set H which is not a *proper* subset of any connected subset of H is called a *component* of H.

components of $K \cdot M_{nij} \cdot I(C'_{np_{ni}})$. By the Zermelo postulate*, there exists a set of points $[P_{nij}]$ such that each set K_{nij} contains just one point P_{nij} and each point P_{nij} belongs to just one component K_{nij}. In the set $[P_{nij}]$ there is a finite subset,

$$P^1_{nij},\ P^2_{nij},\ P^3_{nij},\ \cdots,\ P^{k_1}_{nij},\dagger$$

such that every point of $[P_{nij}]$ is within a distance δ_1 of some point of this finite set. There exists an arc α^r_{nij} $(1 \leqq r \leqq k_1 - 1)$ with end points P^r_{nij} and P^{r+1}_{nij} and lying wholly in M_{nij}. There exists a finite subset,

$$P^{k_1+1}_{nij},\ P^{k_1+2}_{nij},\ \cdots,\ P^{k_2}_{nij},$$

of the set $[P_{nij}]$ such that every point of $[P_{nij}]$ is within a distance δ_2 of some point of $P^1_{nij}, P^2_{nij}, \cdots, P^{k_2}_{nij}$. Let α^r_{nij} $(k_1 \leqq r \leqq k_2 - 1)$ be an arc of M_{nij} with end points P^{r+1}_{nij} and some point of $\sum_{s=1}^{k_1} P^s_{nij}$. Continue this process indefinitely except that for $t > n$ we place the additional condition on $\alpha^r_{nij} (k_t \leqq r \leqq k_{t+1} - 1)$ that it be of diameter less than $\epsilon/(2t)$. This can be done since any two points of K whose distance from one another is less than δ_t can be joined by an arc of M whose diameter is less than $\epsilon/(2t)$.

For each n, i and j, there is a countable set of arcs of M, $\alpha^1_{nij}, \alpha^2_{nij}, \alpha^3_{nij}, \cdots$, such that (a) each lies interior to a hypersphere of radius ϵ/n with a point of K as center, (b) only a finite number are of diameter greater than a given positive number, and (c) each has its end points on K. For each value of n the numbers i and j range over finite sets of values; hence the set of all arcs $[\alpha^r_{nij}]$ for a fixed value of n satisfy conditions (a), (b), and (c) above. And since all arcs $[\alpha^r_{nij}]$ for a fixed value of n are of diameter less than $2\epsilon/n$, the set of all arcs $[\alpha^r_{nij}]$ for all values of n satisfies the condi-

* E. Zermelo, *Untersuchung über die Grundlagen der Mengenlehre*, Mathematische Annalen, vol. 65 (1908), pp. 261–281.

† The symbol k_1 denotes a positive integer whose value depends on n, i and j.

tion that only a finite number are of diameter greater than a given positive number. Let

$$L = \sum_{r}^{\alpha_{nij}}, \quad \begin{matrix} 1 \leqq i \leqq n', & 1 \leqq j \leqq m_n, \\ 1 \leqq r < \infty, & 1 \leqq n < \infty. \end{matrix}$$

We have shown that L satisfies conditions (2) and (3) of our theorem. It remains to prove that (1) and (4) are satisfied. Since only a finite number of the arcs of L are of diameter greater than a given positive number and each has a point on the closed set K, every limit point of L which does not belong to L belongs to K. Thus $K+L$ is closed. Let P be any point of $K+L$. If P does not belong to K it is easy to see that $K+L$ is connected im kleinen at P, for the interiors of hyperspheres of sufficiently small radii and center P contain no point of K and points of only a finite number of arcs of L. If P is a point of K and η is any positive number, there is a hypersphere $C_{np_{ni}}$ which lies entirely in the interior of the hypersphere with radius $\eta/4$ and center P and such that $I(C'_{np_{ni}})$ contains P. Let M_{nij} be the component of $M \cdot I(C_{np_{ni}})$ containing P. There exists a positive number γ such that every point of K whose distance from P is less than γ lies in M_{nij}. There exists a number $\rho > 0$ such that every point p' of L whose distance from P is less than ρ lies on an arc $\alpha_{p'}$ of L, one of whose points e belongs to $K \cdot M_{nij}$ and such that the subarc $p'e$ of $\alpha_{p'}$ is of diameter less than $\eta/2$.* Let σ be the smaller of γ and ρ. Now let Q be any point of $K+L$ whose distance from P is less than σ. If Q belongs to K it belongs to M_{nij}. By the method of

* If S_γ and S_d denote hyperspheres with center P and radii γ and d respectively, then only a finite number of arcs of L have points in $I(S_d)$ for any $d < \gamma$ and contain no point of $I(S_\gamma) \cdot K$ since any such arc is at least of diameter $\gamma - d$. There is a number $d_1 > 0$ such that for $d \leqq d_1$ there is no such arc. Also there is a number $d_2 > 0$ such that no arc of L of diameter greater than $\eta/2$ contains a point whose distance from P is less than d_2 unless the arc contains P. On each of the finite set of arcs of L of diameter greater than $\eta/2$ that contain P there is a point q such that the subarc qP of the arc is of diameter less than $\eta/2$. Let d_3 be the smallest of the finite set of distances from P to the points q. Let ρ be the smallest of the numbers d_1, d_2 and d_3.

construction of L, there is a subset L' of L such that M_{ni} contains L' and $L'+K\cdot M_{ni}$ is connected. But every point of $L'+K\cdot M_{ni}$ is at a distance from P less than $\eta/2$ and $L'+K\cdot M_{ni}$ contains both P and Q. If Q is not a point of K, it lies on an arc α_Q of L which contains a point e of $K\cdot M_{ni}$ such that the subarc eQ of α_Q is of diameter less than $\eta/2$. Then $\alpha_Q+L'+K\cdot M_{ni}$ is a connected subset of $L+K$ containing P and Q and such that every point is at a distance from P less than η. Therefore $K+L$ is connected im kleinen at every point P.

Let P be any point of K at which K is non-dense. Then if S_1 is any hypersphere with center P, the set $I(S_1)$ contains a hypersphere S_2 such that $S_2+I(S_2)$ contains no point of K. Since only a finite number of the arcs of L are of diameter greater than a given positive number, there are only a finite number of arcs of L that have points in $I(S_2)$. Then there is a hypersphere S_3 lying in $I(S_2)$ such that $I(S_3)$ contains no point of L. Then the interior of every hypersphere S_1 with center at P contains a hypersphere S_3 such that $I(S_3)$ contains no point of $K+L$. Hence $K+L$ is non-dense at the point P.

THEOREM 2. *In order that a continuum M lying in euclidean space of n dimensions be a continuous curve it is necessary and sufficient that for every two distinct points A and B of M there should exist a subset of M which consists of a finite number of continuous curves and which separates A and B in M.*

PROOF. The condition is necessary. Let d be the distance from A to B. Let S_1 and S_2 be hyperspheres with center A and radii $d/2$ and $d/4$ respectively. Let $H=S_1+I(S_1)-I(S_2)$. The set $M\cdot H$ is closed and it is easy to see that there is at least one component of $M\cdot H$ containing points on both S_1 and S_2. As M is a continuous curve there cannot be more than a finite number of such components. Let K_1, K_2, K_3, $\cdots$, K_m denote the set of all components of $M\cdot H$ which contain a point on S_1 and a point on S_2. By Theorem 1, for each i, $1\leqq i\leqq m$, there is a continuous curve M_i which contains

K_i, is a subset of M and such that every point of M_i is within a distance $d/8$ of some point of K_i. Suppose that A and B lie in a connected subset of $M - \sum_{i=1}^{m} M_i$. Then there is an arc with end points A and B lying in $M - \sum_{i=1}^{m} M_i$.* This arc contains a subarc α which is a subset of H and has one end point on S_1 and the other on S_2. Then α must belong to some set K_i and thus to $\sum_{i=1}^{m} M_i$. But this is impossible, for $M - \sum_{i=1}^{m} M_i$ contains α. Therefore $\sum_{i=1}^{m} M_i$ separates A and B in M.

The condition is sufficient. If M is not a continuous curve there exist two concentric hyperspheres S_1 and S_2 and an infinite set of subcontinua M_∞, M_1, M_2, M_3, $\cdots$ of M satisfying the conditions of the Moore-Wilder lemma.† Let S_3 and S_4 be distinct hyperspheres concentric with S_1 and lying between S_1 and S_2. Each continuum M_i contains a subcontinuum K_i which contains a point P_i on S_3 and a point Q_i on S_4 and is a subset of the set G consisting of S_3 and S_4 and all points which lie between S_3 and S_4. There exists a sequence of integers $n_1, n_2, \cdots$, such that $[P_{n_i}]$ has a sequential limit point A and $[Q_{n_i}]$ has a sequential limit point B. By hypothesis there exists a finite set of continuous curves $C_1, C_2, C_3, \cdots, C_m$ which are subsets of M and separate A and B in M.

CASE I. Suppose infinitely many of the continua K_{n_i} contain a point of $\sum_{k=1}^{m} C_k$. As there are but a finite number of the curves C_k, one curve $C_{k'}$ must contain a point p_{n_i} of K_{n_i} for infinitely many values of i. The set $[p_{n_i}]$ has a limit point P, which must belong to M_∞ and to G. Let ϵ be a positive number such that no point of $S_1 + S_2$ is within a distance ϵ of P. As $C_{k'}$ is a continuous curve, the point P

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* R. L. Moore, *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922), pp. 254–260. Moore's theorem is stated for two dimensions, but the extension to n dimensions is obvious.

† R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, this Bulletin, vol. 29 (1923), p. 296; R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 371. The lemma holds equally well for n dimensions and for unbounded continua.

belongs to $C_{k'}$ and there is a number $\delta_\epsilon > 0$ such that any point of $C_{k'}$ whose distance from P is less than δ_ϵ can be joined to P by an arc of $C_{k'}$ of diameter less than ϵ. There is a point p_{n_r} of $[p_{n_i}]$ whose distance from P is less than δ_ϵ. Let α denote an arc of $C_{k'}$ with end points P and p_{n_r} and of diameter less than ϵ. The arc α contains a point of M_{n_r} and a point of M_∞ and lies entirely between S_1 and S_2. By the Moore-Wilder lemma, M_{n_r} is a component of the common part of M and the set composed of S_1 and S_2 and all points lying between S_1 and S_2. Hence M_{n_r} contains the arc α. But this contradicts the condition of the lemma that M_{n_r} and M_∞ have no common points.

CASE II. Suppose only a finite number of the continua K_{n_i} contain points of $\sum_{k=1}^{m} C_k$. The set $M - \sum_1^m C_k$ is the sum of two mutually separated sets M_A and M_B containing A and B respectively. Every set K_{n_i} which contains no point of $\sum_1^m C_k$ lies wholly in M_A or wholly in M_B. There is an integer j such that for $i \geqq j$, the continuum K_{n_i} contains no point of $\sum_1^m C_k$. Both A and B are limit points of the set $\sum_{i=j}^{\infty} K_{n_i}$. Either infinitely many of the sets K_{n_i} $(i \geqq j)$ belong to M_A or infinitely many belong to M_B. If the first holds then B is a limit point of M_A; under the second possibility the point A is a limit point of the set M_B. In either possibility we have a contradiction since M_A and M_B are mutually separated.

The assumption that M is not a continuous curve leads to a contradiction with the assumed condition in either case. Therefore the condition is sufficient.

It is to be noticed that in the proof of the necessity of the condition in Theorem 2 we showed that the separating continuous curves were bounded. Hence we have the following corollary and theorem.

COROLLARY. *If A and B are points of a continuous curve M lying in euclidean space of n dimensions, there exists a subset of M which consists of a finite number of bounded continuous curves and which separates A and B in M.*

THEOREM 3. *If K_1 and K_2 are any two mutually exclusive and closed point sets, one of which is bounded, then K_1 lies wholly in a finite number of the complementary domains of K_2.*

PROOF. Suppose the contrary is true. Then there exists an infinite sequence D_1, D_2, D_3, $\cdots$ of distinct complementary domains of K_2 each of which contains at least one point of K_1. For each positive integer i, let P_i denote a point of K_1 belonging to D_i. Let H denote the set of points $P_1+P_2+P_3+\cdots$. By hypothesis either K_1 or K_2 is bounded. If K_1 is bounded, then H is bounded because H is a subset of K_1; and if K_2 is bounded, then since H contains at most one point in the unbounded complementary domain of K_2, it readily follows that H is bounded. Hence, in any case, H is bounded; and since it is infinite, it must have at least one limit point P. Since K_1 is closed and contains H, it must contain the point P; and since K_1 and K_2 are mutually exclusive, P must belong to some complementary domain D of K_2. Clearly this is impossible, since P is a limit point of H, and not more than one point of H can belong to D. Thus the supposition that Theorem 3 is not true leads to a contradiction.

THEOREM 4. *In order that a continuum M in a euclidean space E_n of n dimensions should be a continuous curve it is necessary and sufficient that every two mutually exclusive, closed, and bounded subsets of M should be separated in M by the sum of a finite number of subcontinua of M.*

PROOF.* The condition is sufficient. For suppose a continuum M satisfies the condition but is not a continuous curve. Then by the Moore-Wilder lemma† it follows that there exist two different concentric hyperspheres C_1 and C_2 and a countable infinity of mutually exclusive subcontinua of M: W, M_1, M_2, M_3, $\cdots$ such that (1) if D denotes the

* Compare this proof with that given by R. L. Moore for Theorem 1 of his paper, *A characterization of a continuous curve*, loc. cit.

† See reference to the Moore-Wilder lemma above.

n-dimensional domain whose boundary is C_1+C_2, then each of these continua contains at least one point on each of the hyperspheres C_1 and C_2, and each of them, save possibly W, is a component of the set of points $M\cdot(D+C_1+C_2)$, and (2) W is the sequential limiting set of the sequence of continua $M_1, M_2, M_3, \cdots$. Let A and B denote the sets of points $W\cdot C_1$ and $W\cdot C_2$ respectively and, for each positive integer i, let a_i denote the set of points $M_i\cdot C_1$ and b_i the set $M_i\cdot C_2$. Since A and B are mutually exclusive, closed, and bounded subsets of M, by hypothesis there exists a subset L of M such that (1) $M-L$ is the sum of two mutually separated point sets M_a and M_b containing A and B respectively, and (2) L is the sum of a finite number of continua $L_1, L_2, L_3, \cdots, L_m$. Since neither A nor B has a point in common with L, and A contains no point of M_b and B contains no point of M_a, therefore there exist open sets C_a and C_b, containing A and B respectively, such that C_a contains no point of $L+M_b$ and C_b contains no point of $L+M_a$. There exists an integer δ such that, for every j greater than δ, the point set a_j lies wholly in C_a and the point set b_j lies wholly in C_b. Thus, for every j greater than δ, M_j contains a point of M_a and also a point of M_b. But M_j is a subcontinuum of M, and every subcontinuum of M which contains a point of each of the sets M_a and M_b must contain at least one point of L. Hence, for every j greater than δ, M_j contains a point of L, and therefore of some one of the sets $L_1, L_2, \cdots, L_m$. It follows that there exists an integer g and an infinite sequence of distinct positive integers $t_1, t_2, t_3, \cdots$ such that, for every j, L_g contains at least one point in common with M_{t_j}. Since, for every j, the subcontinuum L_g of M contains a point of M_{t_j} and a point of $M_{t_{j+1}}$ it follows by a lemma of R. L. Moore's* that L_g must contain a point either of a_{t_j} or of b_{t_j}. Thus there exists an infinite sequence of distinct integers $j_1, j_2, j_3, \cdots$, such that either L_g has a point in common with each point set

* *A characterization of a continuous curve*, loc. cit., Lemma 2.

of the sequence $a_{i_1}, a_{i_2}, a_{i_3}, \cdots$, or it has at least one point in common with each point set of the sequence $b_{i_1}, b_{i_2}, b_{i_3}, \cdots$. In the first case it readily follows that A contains at least one point of L_g, and in the second case that B contains at least one point of L_g. But $A+B$ is a subset of $M-L$. Thus the supposition that M is not a continuous curve leads to a contradiction.

The condition is also necessary. For let M be any continuous curve in E_n, and let K_1 and K_2 be any two mutually exclusive, closed, and bounded subsets of M. It follows by Theorem 3 that there exists a finite number $D_1, D_2, D_3, \cdots, D_m$ of the complementary domains of K_2 whose sum contains the point set K_1. For each positive integer $i \leqq m$, let B_i denote the boundary of D_i, let H_i be the set of points common to K_1 and D_i, and let $4d_i$ be the minimum distance between the closed sets of points H_i and B_i. For each point P of H_i+B_i, let C_p denote a hypersphere with P as center and radius d_i, and let G_i' be the collection of all the hyperspheres $[C_p]$ for all points P of H_i+B_i. Since K_1+K_2, and hence also H_i+B_i, is bounded, then by the Borel theorem there exists a finite subcollection G_i of the hyperspheres of G_i' such that H_i+B_i is a subset of the sum I_i of the interiors of the collection G_i. Let T_i denote the point set $(D_i+B_i) - I_i \cdot (D_i+B_i)$. Then T_i is closed. Let F_i denote the sum of all the hyperspheres (not including their interiors) of the collection G_i which enclose at least one point of H_i, and let N_i be the sum of all those which enclose at least one point of B_i. Since the least distance between H_i and B_i is $4d_i$, and since the radius of each hypersphere of G_i is d_i, it follows that F_i and N_i are mutually exclusive closed sets whose least distance apart is $>d_i$. Let Q_i denote the collection of all those maximal connected subsets of M which lie wholly in T_i and contain at least one point of each of the sets F_i and N_i. Each element of Q_i is a continuum, and since M is a continuous curve, it follows by the Moore-Wilder lemma that Q_i has just a finite number of elements. Hence Q_i is a finite

collection of mutually exclusive continua L_{1i}, L_{2i}, L_{3i}, $\cdots$, $L_{n_i i}$ which belong to M.

Now let L denote the point set $\sum_{i=1}^{m}\sum_{j=1}^{n_i} L_{ji}$. Then L is the sum of a finite number of mutually exclusive subcontinua of M. Let M_a denote the sum of all those components of $M-L$ which contain at least one point of A, and let M_b denote the point set $M-(M_a+L)$. No point of B belongs to M_a. For if a point X of B belonged to M_a, then* X could be joined in M to some point Y of A by an arc which contains no point of L, and this arc would contain a subarc t which is a subset of some set T_i and which has its end points on F_i and N_i respectively; and the arc t would necessarily be a subset of some continuum of the collection Q_i, contrary to the fact that t contains no point of L. Therefore B must be a subset of M_b. Since M is connected im kleinen and L is closed, it readily follows that M_a and M_b are mutually separated. Hence $M-L$ is the sum of two mutually separated sets M_a and M_b containing A and B respectively, and therefore L separates A and B in M.

THEOREM 5. *In order that a continuum M in a space of n dimensions should be a Menger regular curve† it is necessary and sufficient that every two points of M should be separated in M by some finite subset of M.*

PROOF. The condition is sufficient. Let P be any point of M and ϵ any positive number. Let C_1 and C_2 be two distinct hyperspheres each of which has P as center and is of radius less than $\epsilon/4$. Let D denote the domain between C_1 and C_2, and let K denote the set of points common to $D+C_1+C_2$ and to M. Then K is closed. Now by Theorem 2 it follows

* R. L. Moore, *Concerning continuous curves in the plane*, loc. cit.

† A continuum M is said to be a Menger regular curve provided that for each point P of M and each positive number ϵ there exists an open subset T of M of diameter less than ϵ which contains P and whose M-boundary is finite. The M-boundary of an open subset T of a continuum M is the set of all those points of $M-T$ that are limit points of T. See K. Menger, *Grundzüge einer Theorie der Kurven*, Mathematische Annalen, vol. 95 (1925–1926), pp. 276–306.

that M is a continuous curve. By hypothesis, for each point X of K there exists a finite subset N_x of M which separates X and P in M. For each such point X, the maximal connected subset H_x of $M-N_x$ which contains X is an open subset of M which does not contain P and whose M-boundary is finite (a subset of N_x). Let G_0 denote the collection of sets $[H_x]$ for all points X of K. Since K is closed and bounded, then by the Borel theorem the collection G_0 contains a finite subcollection G which covers K. Let R denote the sum of all the point sets of the collection G. Then K is a subset of R, and R is an open subset of M. Furthermore B, the M-boundary of R, is finite, for R is the sum of a finite number of the sets H_x. Now, supposing that C_1 is within C_2, let A denote the set of all those points of B which lie on or within C_1. Now $R+A$ does not contain P, for P belongs to no set H_x and to no N_x. Let T denote the maximal connected subset of $M-A$ which contains P. It is readily seen that T must lie within C_1. Hence the diameter of T is less than ϵ. The M-boundary of T is finite, because it is a subset of A. Then, since T is an open subset of M, it follows that P is a regular point of M and that M is a Menger regular curve.

That the condition is necessary follows at once from the definition of a Menger regular curve.

THEOREM 6. *If every two points of a continuum M are separated in M by some finite subset of M, then every two mutually exclusive, closed, and bounded subsets of M are separated in M by some finite subset of M.*

PROOF. It follows by Theorem 5 that M is a Menger regular curve. Then by a theorem of Menger's,* it follows that every two mutually exclusive, closed, and bounded subsets of M can be separated in M by some finite subset of M.

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* Loc. cit., Theorem 12.

THE POLAR CURVES OF PLANE ALGEBRAIC CURVES IN THE GALOIS FIELDS*

BY A. D. CAMPBELL

By imitating the proofs in Fine's *College Algebra* (pp. 460–462) and Veblen and Young's *Projective Geometry* (vol. I, pp. 255–256) we can readily show that also in the Galois fields of order p^n (p a prime integer) we have Taylor's expansion

$$f(x+\lambda X, y+\lambda Y, z+\lambda Z)$$
$$\equiv f(x, y, z)+\frac{\lambda}{1!}(f_x' X+f_y' Y+f_z' Z)$$
$$+\frac{\lambda^2}{2!}(f_x' X+f_y' Y+f_z' Z)^{(2)}+\cdots$$
$$+\frac{\lambda^r}{r!}(f_x' X+f_y' Y+f_z' Z)^{(r)}+\cdots+f(X, Y, Z)=0,$$

where $(f_x' X+f_y' Y+f_z' Z)^{(i)}$ is symbolic for an expression containing derivatives of the ith order, and $f(x, y, z)=0$ is an algebraic curve of order n. In the above expansion we must take all the derivatives as though p were not a modulus, cancel out common factors from numerators and denominators, and then set $p=0$.

The rth polar of (X, Y, Z) with respect to $f(x, y, z)=0$ is

$$\frac{1}{r!}(f_x' X+f_y' Y+f_z' Z)^{(r)}=0.$$

In particular the rth polar of $(1, 0, 0)$ is $(1/r!)\partial^r f(x, y, z)/\partial x^r=0$. We suppose first of all that n has the value

$$n=\alpha p^m+\beta p^{m-1}+\cdots+\gamma p^2+\delta p+\epsilon,$$
$$\epsilon\neq 0, \quad p=\epsilon+\zeta, \quad \zeta\neq 0.$$

* Presented to the Society, December 28, 1927.

We can write the polars of (1, 0, 0) by a sort of detached coefficients, underlining the coefficients that have p as a factor, as follows:

$$(1/1!)[n, n-1, n-2, \cdots, \underline{n-\epsilon}, n-\epsilon-1,$$
$$\cdots, \underline{n-p-\epsilon}, \cdots, \underline{n-p^2-\epsilon}, \cdots, 3,2,1] = 0,$$

$$(1/2!)[n(n-1), (n-1)(n-2), \cdots, (n-\epsilon+1)(\underline{n-\epsilon}),$$
$$(\underline{n-\epsilon})(n-\epsilon-1), \cdots, (n-p-\epsilon+1)(\underline{n-p-\epsilon}),$$
$$(\underline{n-p-\epsilon})(n-p-\epsilon-1), \cdots, (n-p^2-\epsilon+1)$$
$$\cdot(\underline{n-p^2-\epsilon}), (\underline{n-p^2-\epsilon})(n-p^2-\epsilon-1), \cdots, 3\cdot 2, 2\cdot 1] = 0,$$

. . . . . .

$$[1/(\epsilon+1)!][n(n-1)(n-2)\cdots(\underline{n-\epsilon}), (n-1)(n-2)\cdots$$
$$\cdot(\underline{n-\epsilon})(n-\epsilon-1), \cdots, (\underline{n-\epsilon})(n-\epsilon-1)\cdots(n-2\epsilon),$$
$$(n-\epsilon-1)(n-\epsilon-2)\cdots(n-2\epsilon)(n-2\epsilon-1), \cdots,$$
$$(\epsilon+1)!] = 0,$$

. . . . . .

$$(1/p!)[n(n-1)\cdots(\underline{n-\epsilon})\cdots(n-p+1)\cdots, p!] = 0,$$

. . . . . .

where $(n-\lambda)\ (n-\lambda-1)\cdots(n-\lambda-i)$ stands for all the terms of the same $(n-\lambda-i-1)$ power, which then have this common factor in their coefficients. From the above polars we see that the ϵth polar has at (1, 0, 0) a tangent having $(\epsilon+1)$-point contact if (1, 0, 0) is not on $f(x, y, z)=0$, otherwise a multiple point of order $\epsilon+1$. The $(\epsilon+1)$th polar, $(\epsilon+2)$th, $\cdots$, $(p-1)$th polar all have multiple points of order $\epsilon+1$ at (1, 0, 0). Similarly the $(p+\epsilon+1)$th polar, $(p+\epsilon+2)$th, $\cdots$, $(2p-1)$th have at (1, 0, 0) multiple points of order $\epsilon+1$; also the $(2p+\epsilon+1)$th polar points of order $(3p-1)$th, $\cdots$, the $(\theta p^i+\cdots+\phi p+\epsilon+1)$th polar points of order $(\theta p^i+\cdots+\phi p+p-1)$th, etc. Moreover we note that if any one of the polar curves that have multiple points at (1, 0, 0) is a curve of degree $\epsilon+1$, then this polar curve is degenerate. Thus for $p=2$, $n=2^2+1$, $\epsilon=1$, we find the 2d

polar is degenerate; for $p=3$, $n=3+1$, $\epsilon=1$, we find again the 2d polar is degenerate.

If $n=\alpha p^m+\beta p^{m-1}+\cdots+\gamma p^2+\delta p$, i.e. $\epsilon=0$ in n, then all the polars of (1, 0, 0) pass through (1, 0, 0) whether or not this point lies on $f(x, y, z)=0$.

If $n<p$ we find no peculiarities like the above.

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THE CHARACTERISTIC EQUATION OF A MATRIX*

BY E. T. BROWNE

1. *Introduction.* Consider any square matrix A, real or complex, of order n. If I is the unit matrix, $A-\lambda I$ is called the *characteristic matrix* of A; the determinant of the characteristic matrix is called the *characteristic determinant* of A; the equation obtained by equating this determinant to zero is called the *characteristic equation* of A; and the roots of this equation are called the *characteristic roots* of A. If A happens to be a matrix of a particular type certain definite statements may be made as to the nature of its characteristic roots. For example, if A is Hermitian its characteristic roots are all real; if A is real and skew-symmetric, its characteristic roots are all pure imaginary or zero; if A is a real orthogonal matrix, its characteristic roots are of modulus unity. However, if A is not a matrix of some special type, no general statement can be made as to the nature of its characteristic roots. In 1900 Bendixson† proved that if $\alpha+i\beta$ is a characteristic root of a real matrix A, and if $\rho_1\geqq\rho_2\geqq\cdots\geqq\rho_n$ are the characteristic roots (all real) of the symmetric matrix $\frac{1}{2}(A+A')$, then $\rho_1\geqq\alpha\geqq\rho_n$. The extension to the case where the elements of A are com-

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† Bendixson, *Sur les racines d'une équation fondamentale*, Acta Mathematica, vol. 25 (1902), pp. 359–365.

plex was made by Hirsch* in 1902. In 1904 Bromwich† further extended the theorem as follows: If $\alpha+i\beta$ is a characteristic root of a matrix A whose elements are real or complex, and if $\rho_1, \rho_2, \cdots, \rho_n$ are the characteristic roots (all real) of $\frac{1}{2}(A+\overline{A}')$ and $i\mu_1, \cdots, i\mu_n$ are the characteristic roots of $\frac{1}{2}(A-\overline{A}')$, then α lies between the greatest and the least of $\rho_1, \cdots, \rho_n$, and $|\beta|$ does not exceed the greatest of $|\mu_1|, \cdots, |\mu_n|$.

In some cases the theorems just cited give very good limits for the characteristic roots of a matrix, while in other cases the limits are not so restricted. Thus in the case of a real orthogonal matrix these theorems may merely state that the characteristic roots lie in the square $x=\pm 1$, $y=\pm 1$. In this paper we shall give a criterion which in some cases, notably in the case of a real orthogonal matrix, give more restricted limits than the theorems above.

2. *Reduction of a Matrix to a Semi-Unitary Form.* Let A be any square matrix of order n. Then $A\overline{A}'$ is Hermitian and there exists a unitary matrix κ (that is, $\kappa\bar{\kappa}'=I$) such that

$$\kappa A\overline{A}'\bar{\kappa}' = M,$$

where M‡ is zero except in the diagonal, and the elements in the diagonal are the (real) characteristic roots $\rho_1, \rho_2, \cdots, \rho_n$ of $A\overline{A}'$. We may write

$$M = \kappa A\bar{\kappa}'\kappa\overline{A}'\bar{\kappa}' = B\bar{B}', \tag{1}$$

where

$$B = \kappa A\bar{\kappa}'. \tag{2}$$

From (1) the elements b_{ij} of B evidently satisfy the conditions

$$\sum_{t}^{1,\cdots,n} b_{it}\bar{b}_{jt} = \rho_i\delta_{ij}, \quad (i, j = 1, \cdots, n), \tag{3}$$

* Hirsch, Acta Mathematica, vol. 25 (1902), p. 367.

† Bromwich, *On the roots of the characteristic equation of a linear substitution*, Acta Mathematica, vol. 30 (1906), pp. 295–304.

‡ Hilton, *Homogeneous Linear Substitutions*, Oxford, 1914, p. 41.

where δ_{ij} is the Kronecker symbol, and equals 1 if $i=j$; 0 if $i\neq j$. In view of the conditions (3) we shall say that B is in a *semi-unitary* (*semi-orthogonal*, if B is *real*) form. If $\rho_i=1$, $(i=1,\cdots, n)$, B is unitary. We may then state the following theorem.

THEOREM I. *If A is any square matrix of order n there exists a unitary matrix κ such that $\kappa A\bar{\kappa}'=B$, where B is in a semi-unitary form.*

If M is of rank r, κ may be so chosen that $\rho_i>0$, $(i=1,\cdots, r)$; $\rho_i=0$, $(i=r+1,\cdots, n)$. Since $\rho_i=\sum_t b_{it}\bar{b}_{it}=0$, $(i=r+1,\cdots, n)$, evidently $b_{it}=0$, $(i=r+1,\cdots, n;\ t=1, \cdots, n)$; that is, the last $n-r$ rows of B consist entirely of zeros, so that B is of rank at most r. Hence, B must be of rank exactly r. Since the rank of A equals the rank of B, and the rank of $A\overline{A}'$ equals the rank of M, incidentally we have given a proof of the following well known theorem.

THEOREM. *If A is any square matrix of order n, the ranks of A and $A\overline{A}'$ are the same.**

3. *The Characteristic Roots of $A\overline{A}'$.* Referring to the matrix B defined as in (1) and (2), let us form a non-singular matrix $C=(c_{ij})$ by replacing the zeros in the last $n-r$ rows of B by elements $(x_{s1}, x_{s2},\cdots, x_{sn})\neq(0, 0,\cdots, 0)$, such that

$$(4)\qquad \sum_{t}^{1,\cdots,n} b_{it}\bar{x}_{st}=0, \qquad (i=1,\cdots, r;\ s=1,\cdots, n-r),$$

and, moreover, such that

$$\sum_{t}^{1,\cdots,n} x_{it}\bar{x}_{jt}=0, \qquad (i, j=1,\cdots, n-r;\ i\neq j).$$

Thus, we may find $(\bar{x}_{11}, \bar{x}_{12},\cdots, \bar{x}_{1n})$ by determining a non-zero solution of the $n-r$ linear homogeneous equations (4). Having obtained $(x_{11},\cdots, x_{1n})$ we may proceed to find

* Hilton, *Homogeneous Linear Substitutions*, Exercise 4, p. 51.

$(\bar{x}_{21}, \bar{x}_{22}, \cdots, \bar{x}_{2n})$ by adjoining to the system (4) the additional linear homogeneous equation

$$\sum_{t}^{1,\cdots,n} x_{1t}\bar{x}_{zt} = 0 ;$$

and so on. If $\sum_{t}^{1,\cdots,n} c_{it}\bar{c}_{it} = \rho_i$, $(i = 1, \cdots, n)$, then $\rho_i > 0$ and if we write

$$\chi_{ij} = \frac{c_{ij}}{(\rho_i)^{1/2}}, \qquad (i, j = 1, \cdots, n),$$

the matrix χ thus obtained is a unitary matrix. It is evident from the manner in which χ was built up that $B\bar{\chi}'$ is zero except in the diagonal. The elements in the last $n-r$ places in the diagonal are also zero, while those in the first places are $(\rho_i)^{1/2}$, the square roots of the characteristic roots of $A\bar{A}'$. Since $B\bar{\chi}'$ is real and symmetric, the characteristic roots of

$$N = \chi\bar{B}'B\bar{\chi}' = (B\bar{\chi}')^2$$

are the squares of the characteristic roots of $B\bar{\chi}'$, and are therefore the characteristic roots of $A\bar{A}'$. But

$$N = \chi\bar{B}'B\bar{\chi}' = \chi\kappa\bar{A}'\bar{\kappa}'\kappa A\bar{\kappa}'\bar{\chi}' = \chi\kappa\bar{A}'A\bar{\kappa}'\bar{\chi}' = \psi\bar{A}'A\bar{\psi}',$$

where ψ is the unitary matrix $\chi\kappa$. Thus it follows* that the characteristic roots of $\bar{A}'A$ are the same as those of N and therefore of $A\bar{A}'$. Hence we have the following theorem.

THEOREM II. *If A is any square matrix of order n the characteristic roots of $A\bar{A}'$ are the same as the characteristic roots of $\bar{A}'A$.*

Since the unitary matrices κ, χ above are such that

$$\kappa A\bar{\kappa}' = B, \text{ and } B\bar{\chi}' = \chi\bar{B}',$$

it follows at once that

$$\kappa A\bar{\kappa}'\bar{\chi}' = B\bar{\chi}' = \chi\bar{B}' = \chi\kappa\bar{A}'\bar{\kappa}'.$$

Hence

* Hilton, *Homogeneous Linear Substitutions*, p. 20.

$$\bar{\kappa}'\bar{\chi}'\kappa A\bar{\kappa}'\bar{\chi}'\kappa = \overline{A}'.$$

Writing $\bar{\kappa}'\bar{\chi}'\kappa = \phi$, we have the following theorem.

THEOREM III. *If A is any square matrix of order n there exists a unitary matrix ϕ such that*

$$\phi A\phi = \overline{A}'. \tag{5}$$

In this connection compare Hilton, *Homogeneous Linear Substitutions*, Ex. 6, p. 124.

Since from (5)

$$A\phi = \bar{\phi}'\overline{A}' = (\overline{A\phi})',$$

$A\phi$ is Hermitian, so that we have the following theorem.

THEOREM IV. *If A is any square matrix of order n, there exists a unitary matrix ϕ such that $A\phi$ is Hermitian.*

4. *The Characteristic Roots of A.* From (2) the characteristic roots of A are evidently the same as the characteristic roots of B. Suppose then that λ is a characteristic root of B so that there exists a set $(x_1, x_2, \cdots, x_n) \neq (0, 0, \cdots, 0)$ such that

$$\sum_{t}^{1,\cdots,n} b_{ti}x_t = \lambda x_i, \qquad (i = 1, \cdots, n). \tag{6}$$

Taking the conjugates of both members of each of these equations, we have

$$\sum_{s}^{1,\cdots,n} \bar{b}_{si}\bar{x}_s = \bar{\lambda}\bar{x}_i, \qquad (i = 1, \cdots, n). \tag{7}$$

Multiplying corresponding equations in (6) and (7), member for member, and summing as to i, we find

$$\sum_{s,t}^{1,\cdots,n}\left[\sum_{i}^{1,\cdots,n} b_{ti}\bar{b}_{si}\right]x_t\bar{x}_s = \lambda\bar{\lambda}\sum_{i}^{1,\cdots,n} x_i\bar{x}_i;$$

that is

$$\sum_{i}^{1,\cdots,n} \rho_i x_i\bar{x}_i = \lambda\bar{\lambda}\sum_{i}^{1,\cdots,n} x_i\bar{x}_i.$$

Let G be the largest and s the smallest of the characteristic roots of $A\overline{A}'$. Then

$$\lambda\bar{\lambda}\sum x_i\bar{x}_i \leqq G\sum x_i\bar{x}_i,$$

so that $\lambda\bar{\lambda}\leqq G$. Similarly, $\lambda\bar{\lambda}\geqq s$; i. e.,

$$s \leqq \lambda\bar{\lambda} \leqq G.$$

In particular, if A is unitary so that $A\overline{A}'=I$, then $G=s=1$, so that $1\leqq\lambda\bar{\lambda}\leqq 1$; i.e., $\lambda\bar{\lambda}=1$, as is well known. Hence we have the following theorem.

THEOREM V. *If λ is a characteristic root of a square matrix A and if G and s are respectively the largest and the smallest characteristic roots of $A\overline{A}'$, then*

$$s \leqq \lambda\bar{\lambda} \leqq G.$$

THE UNIVERSITY OF NORTH CAROLINA

LIE'S WORKS, VOLUME VI

Sophus Lie: Gesammelte Abhandlungen. Edited by Friedrich Engel and Poul Heegaard. Volume VI, edited by Friedrich Engel. Leipzig, Teubner and Oslo, Aschehoug, 1927. xxiv+940 pp.

In this edition of the works of Sophus Lie his published memoirs are to appear in six volumes while a seventh is to be devoted to supplementary matter. Volumes I and II are to contain those of his memoirs in which the geometrical elements predominate, while volumes III and IV are given to differential equations and volumes V and VI to transformation groups. Volumes III and V have previously appeared; and brief reviews of them are to be found in this Bulletin (vol. 29 (1923), pp. 367–369, and vol. 31 (1925), pp. 559–560, respectively). Volume VI is the third to be published, and it is to be followed by volume IV, on which work is already in progress.

In each of the three divisions the arrangement of the material is mainly chronological. The matter in the first of two related volumes is that which was first published at Christiania while the second is composed principally of the memoirs which appeared in Mathematische Annalen and the publications of the Leipzig Akademie. In this way one avoids having in a single volume two memoirs one of which is mainly a reworking of the other.

The editor's preface of sixteen pages gives a brief analysis of the contents of the present volume VI. In volumes V and VI together one has all the memoirs of Lie on both finite and infinite continuous transformation groups. In these volumes have also been included many works having to do with the applications of the theory of transformation groups to the theory of differential equations. In these Lie sets forth the point of view, with a development of its consequences, from which he approached the theory of transformation groups and their use for the treatment of differential equations. His remaining memoirs on differential equations are assigned to volumes III and IV.

The memoirs on finite continuous transformation groups have for the most part been worked into Lie's large three-volume treatise on the theory of transformation groups (1888–1893), where a nearly complete development has been given in a systematic treatment. The memoirs on infinite continuous groups have not been incorporated in a systematic development, but here the treatment in the memoirs themselves is sufficiently systematic largely to offset this deficiency.

The editor's carefully prepared analysis (pp. vii–xxii) of the contents of this volume relieves the reviewer of the necessity of such a discussion, particularly since that analysis is more adequate than anything which could be said here in the space which might appropriately be used for such a purpose.

The extreme care with which the editorial work has been done—evidently a labor of love—is again shown by the extent of the notes (pp. 755–940); these clear up many points of difficulty and sometimes give

important matter relating to the development of the theories in Lie's mind. Especially useful in this respect is the account given (pp. 777–793) of the antecedents of memoirs II and III on differential invariants and the differential equations which admit a finite continuous group. This account is mainly in the form of excerpts from letters from Lie to Mayer and Klein. Noteworthy is Lie's insistence (p. 781) on the importance of the synthetic element in the origin of his discoveries. Attention may also be called to the account which Lie gives (pp. 781–782), in a letter to Mayer, of the order in which his memoirs may best be studied.

A free translation of a passage (pp. xxi–xxii) from the editor's introduction will not be without interest:

"If one should go through the whole history of mathematics, I believe that he will not find a second case where, from a few general thoughts, which at first sight do not appear promising, has been developed so extensive and wide-reaching a theory. Considered as an edifice of thought Lie's theory is a work of art which must stir up admiration and astonishment in every mathematician who penetrates it deeply. This work of art appears to me to be a production in every way comparable with that . . . of a Beethoven . . . . It is therefore entirely comprehensible if Lie . . . was embittered that 'deren Wesen, ja Existenz, den Mathematikern fortwährend unbekannt zu sein scheint' (p. 680). This deplorable situation, which Lie himself felt so keenly, exists no longer, at least in Germany. In order to do whatever lies in my power to improve the situation still further, ... I have sought to clarify all the individual matters (Einzelheiten) and all the brief suggestions in these memoirs."

R. D. CARMICHAEL

BORN ON ATOMIC DYNAMICS

Problems of Atomic Dynamics. I. *The Structure of the Atom* (20 lectures). II. *The Lattice Theory of Rigid Bodies* (10 lectures), by Max Born.

(Delivered at the Massachusetts Institute of Technology in 1925–26, and published by the Institute.) 8vo., 200 pp.

The name of the author of this volume is alone sufficient to insure an authoritative presentation of the subjects which it treats. As a synopsis—followed rather closely—of a course of thirty lectures, it has the faults and the merits which such a method of development involves. Each lecture is, to a certain extent, devoted to some special phase of the subject, and this allows the reader to find out easily what the author has to say on any topic. For anyone who is familiar with the general lines which atomic physics has taken during the last ten years this has decided advantages. On the other hand, it produces some feeling of a lack of connectedness. A mathematician who wishes to learn of the developments which have taken place up to a couple of years back is advised to read first some general descriptive account and then to turn to Professor Born's volume. If he has some previous acquaintance with the mathematical methods of which use is made, he should have a sufficient basis to find out what developments the

mathematical side needs in order to assist in the interpretation of the observed phenomena.

The first attempts to explain the structure of the atom were naturally made with the use of the classical theory of dynamics which involved the continuity of space and time and of all phenomena deduced from them. It is true that mass was discontinuous but its motions were always continuous. As observation became more searching, it was found that classical dynamics was no longer sufficient to correlate the phenomena and the idea of the "quantum," or change by finite as opposed to infinitely small steps, was introduced. Certain relations of classical dynamics which involved energy and momentum were retained; others involving rates of change had to be abandoned. Hence Professor Born gives a brief summary of the Hamilton-Jacobi presentation of classical dynamics and follows it with the limitations required by the quantum hypothesis.

Then follow several lectures on Bohr's atomic work which has had such great success in correlating the numerical values of the wave lengths of the spectral lines of the elements and particularly those of hydrogen. The numerous observers, however, soon overtook the mathematicians, and modifications and limitations had to be introduced, even to division of the quantum. The reply was the introduction of the harmonic oscillator and the mathematical theory of matrices and still later, wave mechanics. The physical definition of the first is left conveniently vague—mathematically it is a coordinate which is a simple harmonic function of the time. This has had a certain degree of success and several lectures are devoted to it. But that it is by no means final appears in the last lecture where the application of general observational calculus, introduced by Wien and the author, is briefly outlined. In the same lecture the latter gives his views of the trend of future developments, stressing particularly the necessity for careful consideration of the observed phenomena in the mathematical developments.

The second series is largely devoted to attempts to deduce the various observed properties of rigid bodies and particularly those of crystals, from the theories of atomic structure set forth in the first series. The reading and comprehension of this series constitutes a liberal education in general physics. The general geometrical basis of the lattices is mathematically quite simple; the complications are introduced in the attempts to apply it to various crystals.

The volume is lacking on the personal side. While the names of those who have contributed are freely mentioned throughout, the occurrence of these names appears to be quite incidental. No references are given, and while there is no index, the brief table of contents gives a fairly good idea of the subjects treated. These statements are not made as criticisms but simply as descriptive of the subject matter and manner. There can be no question concerning its value as a summary of the status of the subject at the time of the delivery of the lectures.

E. W. Brown

BURGESS ON STATISTICS

Introduction to the Mathematics of Statistics. By Robert Wilbur Burgess, Senior Statistician, Western Electric Company. Boston, Houghton Mifflin, 1927. viii+304 pp. $2.50.

The addition of another textbook on statistics to a field already rather crowded is not justified unless the new book has worthwhile advantages. The reviewer feels that the new text by Burgess meets this test. Up to the present time textbooks on statistics have fallen for the most part into two distinct groups: fairly condensed texts which were intended to appeal mainly to students with considerable mathematical training, and the voluminous texts to be used in connection with work in economics. The text under review combines the two types in a relatively small volume.

Although no two trained statisticians would probably agree in all particulars upon a definition of statistics, it is all to the good that the author undertakes a simple and direct explanation of the meaning and need of statistics from the very start, which not only serves to orient the student but also paves the way well for what follows. The summaries at the ends of the chapters also add to the effectiveness of the text.

The author has succeeded remarkably well in covering so much ground without employing the principles of the calculus. However, the reviewer feels that the author should have been consistent and referred elsewhere for some of the lengthy algebraic work (pp. 101–2, 142, 165, 225, 240–42, 245, 250–51) which will tend to distract the attention of the student from the main discussion unless, as will probably happen, the student skips that work. Such work might well have been relegated to the appendix.

There are several places where the author does not say exactly what he surely meant. For example, we are told (p. 33) to use the ratio chart when we wish "to compare *changes* (my italics) rather than absolute magnitudes." Also the author uses "percentages" in several places where "rates" are surely meant; thus, on this same page (p. 33) he says that "equal vertical distances represent equal percentages of change." In another place (p. 34) we are referred to "the downward curve" of a graph which has a positive slope throughout; the reference is, of course, to a decrease in the slope. Again, we are told (p. 133) that "the number of series (to be used to determine an index number) should be large enough . . . but not so large as to cause undue difficulty . . . ."; we know only too well how some students would interpret this advice. It is rather cruel to say (p. 258) "In order to help the student to understand the *fundamental idea* (my italics) of the probable error (of which little has been said so far), let us follow through the proof of the formula for the probable error of the mean." However, most of such slips are not serious, especially in the light of the text just preceding or just following.

Although one of the claims of the author is generality of treatment, rather condensed explanations of the usual work with index numbers and time series are included. The text carries one finally to the theory of correlation of three and four variables.

The space devoted above to criticism of individual excerpts is relatively much greater than they deserve, and is apt to give an erroneous effect. The main impression made upon the reviewer is that the book is very readable and even interesting (rarely true of textbooks on statistics), except possibly for the algebraic passages referred to above. Teachers of courses which cannot, or do not, require any knowledge of the calculus, will probably find in this book the text they have long sought.

C. H. FORSYTH

HUDSON ON CREMONA TRANSFORMATIONS

Cremona Transformations in Plane and Space. By Hilda P. Hudson. Cambridge University Press, 1927. xx+454 pp.

The appearance of a first exhaustive treatise in any field of mathematics is a matter of concern to those who pursue the particular subject. Books of that type frequently determine the trend of mathematical thought and progress for a considerable period. Perhaps no topic in algebraic geometry has been in greater need of such exposition than Cremona transformations. The earlier presentations are either elementary or incidental to some immediate geometric need. Existing encyclopedic accounts are rather cursory. Thus a large body of researches on the subject, widely distributed in the journals, has been either inaccessible or unknown to those who might wish to become acquainted with the field.

To digest and to unify this mass of material was a task which demanded not merely a mastery of the subject but also an uncommon capacity for detail. This task Miss Hudson has accomplished in a most admirable manner in the book under review.

The book itself gives an impression of unity which is rather remarkable in view of the diversity of the transformations of which it treats. This doubtless is due to the wisdom of the author in selecting from the field a naturally related group of topics. Only transformations in the plane and in space are considered. For each case these are discussed first with reference to the properties common to all and secondly with reference to their division into various types. The single application considered is to the resolution of singularities of curves and surfaces and a treatment of this is practically inevitable since such singularities are present in the transformations. This is the division of the subject to which most of the author's own contributions have been made. No geometric applications are given except as they may be involved in the construction of a type, or as they may be inherent in the general class such as the isologues of a transformation in superposed planes or the complex associated with a transformation in space. Applications to other fields of mathematics are omitted. No account of Cremona groups, finite or infinite, appears.

The nomenclature of the author is on the whole well chosen even though individual contributors to the subject must generally expect to find that their own notations have not, in all cases, been adopted. Miss Hudson adopts the term *F-system* (*fundamental system*) for the aggregate of points in S whose correspondents in S' are indeterminate; and *P-system*

(*principal system*) for the aggregate of points in S whose correspondents in S' are adjacent to (that is, directions about) the points of the F-system in S'. It is to be hoped that this usage may become general. No confusion can arise in the plane, by calling a P-curve an F-curve and this is frequently done but in space a distinction must be made.

About one-third of the text is devoted to the planar transformations, one-half to the spatial transformations, and the remainder to a historical sketch, a bibliography of 425 titles, a series of tables of types, and an index. In the plane, Chapters I and II are devoted to the general theory; Chapter IV to the determination of the types of transformation; Chapter III to the quadratic transformation, and Chapter VI to other special types; and Chapter VII to the resolution of the singularities of plane curves. In Chapter V we find the geometry which arises when the planes S, S' are brought into coincidence; and in Chapter VIII proofs of Noether's theorem that every transformation is a product of quadratic transformations. For space we find in Chapters IX and XIII the general theory; and in Chapters X and XIV the quadratic and other particular transformations. The intervening Chapters XI and XII are devoted respectively to postulation and equivalence, and to contact conditions. In Chapter XV the cubo-quartic transformation, which presents examples of most of the phenomena previously discussed, is minutely examined. Chapter XVI contains an account of the work up to the present time on the resolution of the singularities of surfaces. The final Chapter XVII contains the historical sketch and the bibliography.

In general the exposition is clear, brief, and effective. One may differ at times from the author with respect to methods of proof and emphasis on aspects of the theory. Thus there is associated with a given Cremona transformation T a linear transformation L, with integer coefficients, first freely used by S. Kantor, which expresses the effect of T upon the curves of order n in the plane. L and its invariant linear and quadratic forms are introduced rather incidentally (p. 26). But from the obvious invariance under T of the genus of one curve, and of the free intersections of two curves, there follows the invariance of the two forms under L; and therefore as an immediate consequence the incidence relations of p. 18. Also the nature of the product T_1T_2 (p. 56) is immediately read off from the coefficients of the product L_1L_2. In the determination of types carried out in Chapter IV the emphasis is placed on a division of types with respect to the degree n of T. It is the reviewer's opinion that a division with respect to the number σ of F-points of T is more fundamental. Nevertheless on these two matters, and indeed throughout, the presentation follows the historical order of development.

It is in connection with the space theory that the book will be of greatest service to the advanced reader. The original articles, in widely different notations, with many obscurities and not a few errors, are here combined into an organic whole whose lapses from perfection are as a rule clearly indicated. The content is brought up to present date with, for example, a proof of the recently announced theorem of Tummarello on the equality of the sums of the genera of F-curves in S, S'; and with certain

novelties such as the construction of a T with an arbitrarily given curve as an F-curve of the first kind.

In the chapter on the resolution of singularities of surfaces the author does not give complete proofs but refers on essential points to the sources. There is indeed difference of opinion as to whether such resolution has actually been accomplished. Thus the present exposition of progress to date is opportune.

References to the bibliography are made in the text by number. This convenient method would have been much improved if the authors cited were also given.

A forthcoming report on topics in algebraic geometry contains chapters on Cremona transformations which will cover as well the matters not discussed by Miss Hudson. Fortunately the prior appearance of this volume has enabled the authors concerned both to make their general list of references more complete, and to refer for a broader account to the book itself.

A. B. COBLE

JORDAN ON STATISTICS

Statistique Mathématique. By Charles Jordan. Paris, Gauthier-Villars, 1927. xvii+344 pp.

This book is a scholarly treatment of the subject of mathematical statistics. It is in many respects the book which mathematicians have been waiting for. It performs for statistics much the same service as a standard *Cours d'Analyse* performs for analysis, and with the same typical French clearness; it would seem to be also of about the same degree of difficulty. Moreover, in point of difficulty, it is, considering the variety of subjects treated, of remarkably uniform grade throughout. The development is carefully planned in advance, and is carried out logically, and with uniform notation and nomenclature, producing maximum clearness of exposition in minimum space. There are no topics of dominant importance which the author has not considered, although there are some minor matters—this is true of almost every book—which he has omitted and the reviewer would like to have had included. Apparently the author has also drawn on all the important sources, but though many times it is possible to recognize in his exposition the essential arguments of others, never are these arguments copied bodily; they always bear the impress of his own method of thinking. The only possible criticism in this connection is that, in acknowledging his indebtedness to others, Jordan rather frequently mentions names only, without giving references to the publications—a flaw which could easily be remedied in a second edition.

Although the various chapters are almost uniformly good, the reviewer was particularly impressed with the first three introductory chapters and with those on frequency curves (9 and 10) and less well pleased with the final chapter, on sampling. The first three chapters contain an excellent account of the preliminary notions; the material is well chosen and the sequence carefully planned. Much is left unproved here; the account is

rather descriptive with only an occasional demonstration. Chapter 4, on probability, contains some stimulating material, including a rather elegant account of the series of Lexis and Poisson, along with much that is rather ordinary. The author is not always careful here to distinguish between what is really a definition and what is a theorem. The fundamental theorems which are hardest to demonstrate rigorously from his definition of probability, he does not prove at all, possibly because he makes reference to an account of probability which he has published elsewhere. His "empirical principle" is so vaguely stated that it would seem impossible to use it rigorously in a demonstration: (the italics are mine) "Si l'on répète une expérience *un grand nombre* de fois, chaque évènement se produit *à peu près* en un nombre de fois proportionnel à sa probabilité." On the next page (77) *évènements indépendents* are not defined at all. Assuming the author's definition of probability on page 28, his statement of Bayes' Theorem on page 79 is not really a theorem, but an extension of the meaning of the earlier definition. It is rather unfortunate to say that (p. 86) J. Bernoulli proved his celebrated theorem when his so-called demonstration was really so very bad. Jordan lays a good deal of stress on the usefulness of the normal law as an approximation, but omits the case where the approximation is specially good, viz.: when it is a question of summing an equal number of terms of a point binomial on both sides of the mean. The reviewer gravely doubts that (p. 120) "experience proves that the (normal) law of errors is alwa, s valid when the errors are small relative to the magnitudes measured." Jordan's account of the method of least squares is too compressed to be of mucn service to the reader. It follows classic lines, as do his directions for computation, and so he implies that he is not expecting the computer to use a machine!

Chapter 5, on *Classification Statistique*, is quite similar to Yule's treatment, but enjoys distinction from Yule's in not being verbose. Averages and moments are treated in the sixth chapter. Again there is a refreshing absence of that elaboration of the obvious which is characteristic of so many books at this point. Some of the illustrative material in this chapter is very interesting. Chapter 7 is on the point binomial. Chapters 9 and 10 are on frequency curves, and as has been indicated above, they seem to the reviewer excellent. He does not entirely agree with what appears to be the author's point of view (p. 207) as to the purpose of obtaining these analytical curves, namely, that it is solely to find a background of probability theory which one can then claim determines the data; but, adopting it for the moment, it would appear unfortunate not to have the "*différentes considérations*" (p. 241) which underlay Pearson's differential equation explained. For it is that very background of probability which has given Pearson's curves a theoretical advantage over certain other methods of representation. Jordan is, however, consistent when he criticizes Charlier's series and some others because they give negative frequencies. But the very small negative frequencies which he condemns are nevertheless very good approximations to the very small positive frequencies that they are supposed to equal, and so if one will admit that an important purpose of such series is to substitute for unwieldy expressions simple and manageable

approximative formulas, that particular criticism is answered. Chapter 10, on certain series fitted to frequency distributions by least squares, contains valuable suggestive material. Graduation, in Chapter 11, is treated rather curtly, but it was clearly necessary to avoid long incursions into special fields of application, and therefore it was probably necessary for Jordan to stop where he did or else so to extend this chapter that it would become out of harmony with the others.

Finally, we have the theories of correlation in Chapter 12, and of sampling in Chapter 13. This last subject involves matters of extreme delicacy, and they cannot be adequately set forth in a brief ten pages, even though all the statements be technically accurate. It is too easy here to confound approximations with exact statements, and indeed it is difficult to be sure that so-called approximations are not really wide of the mark. One such approximation which the author fails to indicate as such is "$\delta\mu_2 = 2\zeta\delta\zeta m$," used to find the probable error of the standard deviation when the probable error of the square of the standard deviation is given. Formula (15), which precedes the above equation, is also open to suspicion without the extra terms which have just been dropped. It is of course quite true if one admits the hypothesis that "$\eta = 0$," but this condition is not usually satisfied in the many cases to which the formula and its derivatives are usually applied. Jordan seems to be aware of this himself but does not properly warn the reader.

The book is bound only in paper, but the printing and materials are good. The proof reading has been almost perfect. Only a few misprints have been noted, and they are listed below; no one of them happens to be of such a character as to mislead the reader, except perhaps the one on page 49.

Page 3, line 12, for *Chieff* read Chief. Page 13, first line of footnote, for $n(u)_n$ read $nh(u)_n$. Page 36, last line, the omitted numerator of the first fraction is s. Page 49, in the equation between equation (16) and equation (17), replace 2 by -2. Page 83, line 6 from bottom, for $\binom{6}{7}$ read $\binom{6}{1}$. Page 92, line 11, for 3, read 13. Page 241, line 12 from bottom: the omitted first letter should probably be L. Page 266: the upper limit is omitted in the integral of (1); it should be plus infinity. Page 288, first line of footnote, for *adjustement*, read *adjustment*. The page following page 298 should be numbered 299, instead of 29.

B. H. CAMP

SHORTER NOTICES

The Theory of Integration. By L. C. Young. Cambridge University Press, 1927. vii+52 pp.

This is one of the well known Cambridge Tracts in Mathematics and Mathematical Physics. A detailed discussion of the contents in therefore superfluous, as the chief duty of the author is to present the material intelligibly and accurately; in this case these requirements are satisfied. On the other hand it seems to the reviewer desirable to call attention to two defects of another kind. As the author points out in his preface, "the object of small treatises of this kind is to enable the general student to gain rapid access to the various branches of Modern Mathematics." Viewed in this light the title is somewhat misleading, as the book deals almost exclusively with W. H. Young's theory of integration. Although this is stated in the preface, it is to be feared that the general student would not realize that he had read *a* theory of integration and was still unacquainted with the one most widely known. The other defect is the total absence of references to the literature. Surely a small, well chosen bibliography should be included in any book of this kind.

W. A. Wilson

Integral Bases. By W. E. H. Berwick. Cambridge University Press, 1927. 95 pp.

This new Cambridge tract contains the author's investigations of the problem: *To determine an integral base for an algebraic field given by an arbitrary equation* $f(x)=0$. This problem can be reduced to the determination of a partial base for all primes p dividing the discriminant of $f(x)$. For this case the author has, as he asserts in the preface, evolved various new methods, which can also be used for finding the decomposition of p in prime-ideals. "Failing cases exist, but the approximations given are sufficient to cover nearly any numerical equation not specially constructed to defy them."

The method is, in brief, the following: Let

$$f(x) \equiv \phi_1(x)^{a_1} \cdots \phi_r(x)^{a_r} \pmod{p}$$

be the decomposition of $f(x)$ in prime-functions (mod p); then $p = A_1 \cdots A_r$, where all ideals A are relatively prime. To find the further decomposition of A, we write $f(x)$ in the form

$$f(x) = \sum Q_i(x) p^{\alpha_i} \phi(x)^i,$$

where $Q_i(x) \not\equiv 0 \pmod{p, \phi(x)}$ and construct the Newton polygon to the lattice-points (i, α_i). Then $A = B_1^{\lambda_1} \cdots B_u^{\lambda_u}$, where k is the number of sides and λ_i certain constants. Under certain conditions the prime-ideal decomposition of p and the corresponding partial bases can be found in this way.

The author seems to be unaware of the fact that his new method and his results have already been completely evolved in my paper *Zur Theorie*

der algebraischen Körper, Acta Mathematica, vol. 44 (1923), pp. 219–314, and that this method has been used for the solution of the same problems and various other problems in the theory of algebraic fields (Acta Mathematica, vols. 45–46). I also mention that it is possible to generalize the method so that it can be applied in every case.

The book is difficult to read, but contains various good numerical illustrations of the method. The final chapter is dedicated to the study of fields defined by a binomial equation $x^n-a=0$ and gives an elaborate study of the different possibilities arising in this case.

OYSTEIN ORE

Einführung in die Wahrscheinlichkeitsrechnung. By J. L. Coolidge. German translation by Dr. Friedrich M. Urban. Leipzig, 1927. ix+212 pp. 10 R. M.

In the one-page preface of the translator the reason is given for publishing a German edition of this American work. There are several German texts of a very high order on this subject, but none of them appears to meet the needs of the beginner in such an attractive and elementary manner. The application of statistics to new fields is constantly increasing and many of the methods of the theory of probability must be used by investigators whose principal interest is not in mathematics. They have not had the training necessary for profitable study of the advanced treatises. This justifiable demand ought to be met by a book which has few preliminary mathematical requirements.

To the English student who is beginning to extend his study of probability to the great German works this translation will be the Rosetta stone that furnishes him the required technical vocabulary. This is important when we lack an interlingual mathematical dictionary. Many of the terms used in probability have fine distinctions in our own tongue, such as dispersion, error, deviation and discrepancy. When we meet such words in another language for the first time, it sometimes requires an immoderate amount of reading and re-reading to establish the proper vocabulary.

The translation is quite accurate in the technical statements but some of Coolidge's delightful expressions were not adequately turned. "This unlovely formula" (p. 170) becomes "diese hässliche Formel" (p. 168). With a little more care this might have been rendered by a more appropriate adjective, yet one could hardly expect to find a single word in German carrying the same double meanings as Coolidge's "unlovely." "Unliebenswürdige" would translate one meaning, the other might be given by "unzierliche," which is as unusual in German as "unlovely" is in English.

Most of the errors mentioned in the review by H. L. Rietz of the English edition in this Bulletin (vol. 32, No. 1) have been corrected, but in formula (11), Chapter III, $1/\sqrt{2npq}$ has become $1/\sqrt{2n\pi q}$ where it should be $1/\sqrt{2n\pi pq}$. A few others have also been corrected.

F. A. FORAKER

Conductibilité Électrique des Métaux et Problèmes Connexes. Rapports et Discussions du Quatrième Conseil de Physique, Institut International de Physique Solvay. Paris, Gauthier-Villars, 1927. 368 pp.

This is the proceedings of the Fourth Solvay Conference on conduction in metals held in Brussels in April 1924. It contains nine more or less comprehensive papers presented by leading physicists interested in that subject, and also the discussions by those attending the conference. The papers run in the following order: H. A. Lorentz, *On the application of electron theory to the properties of metals;* P. W. Bridgman, *On the phenomena of metallic conduction in general and some possible theoretical explanations;* O. W. Richardson, *On another proposed theory of metallic conduction;* W. Rosenhain, *On the structure of alloys;* W. Broniewski, *On electrical resistance and thermal expansion;* A. Joffé, *On the conductivity of crystals;* Kamerlingh Onnes, *On some recent researches on superconductivity;* E. H. Hall, *On metallic conduction and the transverse effects in a magnetic field;* A. Joffé and N. Dobronravoff, *On some experimental tests of the idea of light quanta.*

S. C. WANG

Lehrbuch der Variationsrechnung. By Adolph Kneser. Second edition. Braunschweig, F. Vieweg & Sohn, 1925. 397 pp.

In the second edition of his Lehrbuch der Variationsrechnung Kneser has followed, in general, the order of development of the first edition. There is, however, much difference in the two texts, and the second edition is certainly an improvement over the first.

Those familiar with the older edition will recall that while chapter headings were given there were no section headings. Consequently when one wished to know what notion was to be discussed in a given section he had either to read it through or else turn back to the table of contents. It is, therefore, gratifying to turn to the first page and find listed the topic discussed in the first section. It is much more gratifying to find that the method of presentation is improved. The reader need only study any chapter of the new edition and then work through the corresponding chapter of the first edition in order to be impressed by this change.

This fact so impressed itself upon the present reviewer that he decided to look up the comments of the reviewer of the first edition (this Bulletin, vol. 12, p. 172). It was found that while that reviewer classed the work as one of monumental importance, he also asserted that it was not lucid. Most of us who have had occasion to use the first edition agree with this comment. There is no doubt, however, that this criticism has been overcome to a very great extent in the new edition. Here the notation has been explained in greater detail, statements of needed existence theorems inserted at the proper places and the illustrative examples explained more fully. While some of the theoretical developments which have taken place since 1900 have been inserted the work has not been complicated.

This second edition will continue to be recognized as one of the most important texts on the calculus of variations.

E. J. MILES

Examen des Différentes Méthodes Employées pour Résoudre les Problèmes de Géométrie. By G. Lamé. Paris, Hermann (no date). xii+124 pp. 2 plates.

This rather unusual little book will be read eagerly and profitably by the student whose grasp of the rudiments of elementary geometry, algebra, analytic geometry, calculus, and descriptive geometry is sufficient to enable him to appreciate comparisons of their methods, aims, limitations, and applicability. The teacher will be indebted to the author not only for the occasional paragraphs of suggestions on teaching, but also for the help gained indirectly from a book which is something more vital than a mere store-house of information. In the author's opinion, the most noteworthy parts of the book are: the analytic expression of the common intersection of geometric loci; the complete determination of curves and surfaces of the second degree by descriptive geometry, when a sufficient number of points are given; and the theory of curves and surfaces represented by the equations $x^a/a^a+y^a/b^a=1$ and $x^a/a^a+y^a/b^a+z^a/c^a=1$.

E. B. COWLEY

Problems of Modern Physics. By H. A. Lorentz. Boston, Ginn and Co. vi+312 pp.

That a course of lectures delivered at the California Institute of Technology in 1922 by Lorentz now appears in print edited by Bateman is cause for rejoicing. Lorentz is always good reading because he is so sound and so honest. He gives both sides of a question and in our present state of perplexity in physics we need both sides. Take, for instance, his §45 (pp. 134–140) entitled "How Far Can Molecular Forces be Explained by the Electromagnetic Actions?" in which he presents and comments on work by Born and Landé. Or (p. 95) the discussion of the reality of variations of length due to translation. And (p. 47 et seq.) the question of whether moving electrons do radiate. Again §42 (p. 125) commenting on the structure of the electron and (in numerous places) on our troubles with quanta. Throughout, the book is illuminating.

Many years ago Lorentz adopted the plan of composition which develops the main line of thought with relatively simple formulas in the main text and relegates to an appendix the detailed mathematical analysis. This plan is followed here and adds to the readability of the book. Moreover the author has the faculty of getting directly and simply at the fundamental problems he wishes to discuss and thus covers a deal of physics in a brief space and time. He gets into general relativity, including Levi-Civita's parallelism, with no great mathematical splurge. Why cannot more of us write that way? Probably because we cannot think hard enough. It would be desirable for Lorentz to come again and cover the many new ideas which have sprung up since 1922 and which in his brief preface he says might form the subject of a treatise.*

E. B. WILSON

* Since this was written, the scientific world has been saddened by the news of the death of this great man.

Geometria Proiettiva Differenziale. By G. Fubini and E. Čech. Bologna, Nicola Zanichelli, 1927. Vol. II, 406 pp.

The second volume of the Projective Differential Geometry of Fubini and Čech is a direct continuation of the first. Chapters, sections, and pages are numbered consecutively, the second volume beginning with Chapter 8, §71, page 389; and an index is provided at the end of the second volume for the entire work. Moreover, the continuity is not altogether mechanical. In the last chapter of the first volume it was proved that if a non-ruled surface in ordinary space is projectively deformable at all, then it is projectively deformable in ∞^3, ∞^2, or ∞^1 ways, and the concluding sections of that volume were devoted to a discussion of the first case. The opening chapter of the second volume completes the discussion of the other two cases.

Chapters 9 and 10 are devoted to a consideration of various portions of the projective differential geometry of a surface in the neighborhood of one of its points. Among the topics treated are the quadric of Moutard; Čech's transformation Σ between points in a tangent plane of a surface and planes through the contact point, with various special cases one of which is connected with a two-dimensional metric of Weyl; the canonical pencil of lines; and the cone of Segre. Arbitrary curvilinear coordinates are used in these chapters.

Complexes and congruences of lines are studied in Chapter 11. Here the closest possible analogy with the theory of surfaces is preserved. The theory proceeds from certain fundamental differential forms to the differential equations which determine the configuration except for a projective transformation. Much emphasis is placed on the notion of projective applicability of complexes and of congruences.

The projective differential geometry of hyperspace appears for the first time in Chapter 12. The now familiar transition is made from the fundamental differential forms for a hypersurface in S_n to the differential equations, which might themselves have been used as fundamental. Non-parabolic surfaces in S_4 are studied, and ruled surfaces in spaces of even and odd dimensions are briefly considered.

It seems appropriate to remark in this connection that a general projective differential theory of a V_k in an S_n, with $n>4$ and $1<k<n-1$, remains yet to be constructed. Two methods seem to be available for constructing such a theory, namely, the method of Wilczynski who starts with a system of differential equations and employs the Lie theory of continuous groups, and the method of Fubini who starts with a system of differential forms and employs the absolute calculus of Ricci. But the labor involved in the former method seems at present to be prohibitive, and the latter method has so far failed because of the lack of a suitable fundamental quadratic form, or perhaps because of the lack of a suitable absolute calculus for an n-ary p-adic differential form.

Nearly the last third of the second volume is occupied by four appendices written by three different authors. In the first appendix Tzitzeica treats in French a special problem in the deformation of surfaces. In the second Bompiani gives a systematic exposition of his own contributions

to the projective differential geometry of curves and surfaces in ordinary space. Bompiani has been especially interested in geometrical definitions of the fundamental forms of Fubini. In the third appendix Terracini gives an exposition of the most interesting results in the projective differential geometry of hyperspace, accompanied by a valuable bibliography, and in the fourth the same author gives a brief account of a special problem concerning surfaces whose asymptotics belong to linear complexes.

These volumes constitute the only existing comprehensive treatise on the modern theories of projective differential geometry. Every one interested in this subject will need to have a copy of this work available, as references will be made to it for some time to come. There still seems to be room, however, for a well organized treatment in English of the same material. And most of all there is need for a text which can be put into the hands of graduate students who are being introduced to the subject, because the monumental work of Fubini and Čech, valuable as it is for the expert, does not seem to be suited to the needs of a beginner in this field.

E. P. LANE

Les Mathématiques du Chimiste. By L. Gay. Paris, J. Hermann, 1926. vii+208 pp.

As indicated by the title, this book is not a mathematics text, but rather a monograph embodying only those principles which the author believes to be of particular utility to the chemist; more especially to the physical chemist. Hence he has omitted all reference to trigonometric functions, polar coordinates, integrals of surfaces and volumes, the mathematics used in crystallography (believing this to belong to the realm of physics), and he has only touched on the subjects of limits, series, imaginaries, etc.

The scope of the book may be judged from the following consecutive chapter headings: powers and roots, logarithms, functions of a single variable, functions of more than one variable, differentials, derivatives of functions of more than one variable, integrals of functions of a single variable, and integrals of differentials which depend upon more than one variable.

It seems to the reviewer that Chapter one might have been omitted; surely every chemist must be familiar with its contents. Much of Chapter two, dealing with logarithms, might also have been deleted, especially the discussion of their use in multiplying, dividing, raising to powers, and extracting roots. However, the reviewer is pleased to find included in the chapter an explanation of the numerical values of logarithms of numbers less than one, and a discussion of the errors which may result from the use of logarithms in making computations. In Chapter three it seems scarcely necessary to explain how to plot using rectangular coordinates, or to point out that the graph of a first order equation is a straight line. Methods of interpolation and extrapolation are given the attention they well deserve. In Chapter four we find a goodexposition of the use of the triangle in representing systems of three components. In Chapter five are given the rules

for differentiating the more common algebraic equations. Although the author has defined the use of d and Δ in the usual way, he seems at times to confuse the two conventions, especially in his discussion of the differential of the product of two variables.

Chapters six and seven contain a brief treatment of first and second derivatives, series, the characteristics of maxima and minima, points of inflection, and first and second partial derivatives. In Chapter six it seems unfortunate that the author chose to discuss van der Waal's equation especially as applied to gases in the neighborhood of their critical points which is just the region in which the equation gives the most unsatisfactory results. The last two chapters deal with integration. Here we find that it is quite simple to integrate an irregular area by cutting out the plotted area, weighing the paper, and comparing the weight with that of a paper of the same thickness and quality and of known area. Although this method has been repeatedly recommended, it is the experience of the reviewer that it may lead to considerable error, and that it should be used with considerable skepticism. The characteristics of exact differentials and their physical significance are also discussed.

The book contains some errors, presumably typographical. On page 42, in discussing the velocity of a first order reaction, we read that the temperature, instead of the time, is a logarithmic function of the concentration. On page 59, heat of fusion should read heat of vaporization. On page 68, the corrected chemical equation is $2BaO+O_2 \rightleftarrows 2BaO_2$. In the footnote on page 78 the name epicycloid is given to the curve usually considered as the cycloid. In the footnote on page 128 lead is, erroneously, said to crystallize; this should read bismuth. On page 199, in the integration between the limits A and B it would seem more logical to use the subscripts A and B for the variables in the final equation. On page 205,

$$\frac{-l}{T_2} \text{ should be } \frac{-l}{T^2}.$$

In the opinion of the reviewer, the value of the book would have been enhanced if the author had included equations for the common curves, more about empirical equations, something on the method of least squares, and if he had laid more emphasis on the errors in results calculated from experimental data.

The book is evidently intended primarily for French consumption. The author in his preface states that there is no other book of its scope in the French language. This, of course, cannot be said for the English language. However, the English student of chemistry will find much of value in this little book. Its contents is common knowledge to the research worker in physical chemistry.

BLAIR SAXTON

NOTES

The opening number of volume 30 of the Transactions of this Society (January, 1928) contains the following papers: *Simpler proofs of Waring's theorem on cubes, with various generalizations*, by L. E. Dickson; *Cubic curves and desmic surfaces*, second paper, by R. M. Mathews; *Possible orders of two generators of the alternating and of the symmetric group*, by G. A. Miller; *Optics in hyperbolic space*, by James Pierpont; *Geodesics on surfaces of genus zero with knobs*, by D. E. Richmond; *Concerning end points of continuous curves and other continua*, by H. M. Gehman; *The apportionment of representatives in Congress*, by E. V. Huntington; *Conditions for associativity of division algebras connected with non-abelian groups*, by John Williamson; *A generalization of Taylor's series*, by D. V. Widder; *A problem in the calculus of variations with an infinite number of auxiliary conditions*, by R. G. D. Richardson; *A contribution to the theory of fundamental transformations of surfaces*, by M. M. Slotnick.

The opening number of volume 50 of the American Journal of Mathematics contains: *Additive number theory for all quadratic functions*, by L. E. Dickson; *Démonstration de quelques propriétés des ensembles abstraits*, by M. Fréchet; *On certain points in the theory of Dirichlet series*, by J. F. Ritt; *On a theorem of Severi*, by O. Zariski; *On Gierster's classnumber relations*, by J. V. Uspensky; *Note on a theorem of Bôcher*, by G. C. Evans; *Generalized Neumann problems for the sphere*, by G. C. Evans; *On Taylor's series admitting the circle of convergence as a singular curve*, by J. J. Gergen and D. V. Widder; *A mathematical theory of depreciation and replacement*, by C. F. Roos.

The opening number of volume 29, series 2, of the Annals of Mathematics (December, 1927) contains: *Developments in Hermite polynomials*, by M. H. Stone; *A generalization of the calculus of finite differences to include the differential calculus*, by J. P. Ballantine; *Relations satisfied by coefficients of periodic solutions*, by W. J. Trjitzinsky; *An integral equation with an associated integral condition*, by L. Guggenbühl; *Partition polynomials*, by E. T. Bell; *A new formulation for general algebra*, by J. W. Young; *A determination of the groups of order p^5*, by H. A. Bender; *Representation of functions determined by their initial values*, by W. J. Trjitzinsky; *An integration method of summing series*, by G. James; *On groups of order p^m, p being an odd prime number, which contain an abelian subgroup of order p^{m-1}*, by H. A. Bender; *Osculating derivative of a ruled surface*, by D. Sun; *Chains of congruences for the numerators and denominators of the Bernoulli numbers*, by J. L. Bell.

A new journal, the Boletin Matematico, has been founded at Buenos Aires, under the editorship of Professor B. I. Baidaff. The first number appeared in January, 1928.

A committee of this Society, consisting of Professors T. S. Fiske (chairman), R. C. Archibald, J. L. Coolidge, L. E. Dickson, E. R. Hedrick, Dunham Jackson, James Pierpont, M. I. Pupin, R. G. D. Richardson, and Oswald Veblen, has been appointed to arrange for fitting activities to mark the fiftieth anniversary of the founding of the Society in 1888.

The Calcutta Mathematical Society will celebrate the twentieth anniversary of its foundation by publishing a Commemoration Volume in the fall of 1928.

At the Nashville meeting of the American Association for the Advancement of Science, the following officers (among others) were elected: president, Professor H. F. Osborn; vice-president of Section A, Professor R. C. Archibald; secretary of Section A, Professor C. N. Moore. Professor L. E. Dickson was elected a member of the Council, and Professors D. R. Curtiss, F. R. Moulton, M. I. Pupin, and Edwin B. Wilson members of the Executive Committee. Professor Oswald Veblen is a member of the Committee on Grants for Research.

Owing to the favorable reception which has been given to the Mémorial des Sciences Mathématiques, a similar series to be known as the Mémorial des Sciences Physiques is to be published under the auspices of the French Academy and several other Academies, under the direction of Professors Henri Villat and Jean Villey. A number of titles of forthcoming volumes have been announced.

The Carnegie Foundation has recently presented the sum of $10,000 to the American Philosophical Association as a revolving fund for the publication of a series of source books in the history of the sciences, under the general editorship of Professor G. D. Walcott of Hamline University. It is proposed that each volume of about 600 pages, shall contain extracts from the most important contributions to science from about 1500 to 1900. At least one of these volumes will relate to mathematics, and will be under the immediate charge of a committee consisting of Professors R. C. Archibald, Florian Cajori, and David Eugene Smith (chairman). The articles will be about 6–20 pages each, including photographs of the original text, a translation, and comments. Numerous suggestions of articles for reprinting have been made already, but further suggestions with precise references to the original articles will be welcomed. Volunteers to assist in preparing and editing specific articles are especially desired. Suggestions and offers of assistance may be sent to any member of the committee.

An informal conference on analysis situs was held at Princeton University on April 4–5, 1928, in order to provide a close contact and a thorough exchange of ideas among those particularly interested, in advance of the symposium on the same topic at the meeting of the Society in New York on April 6–7. Among those participating were Professors Alexander, Lefschetz, and Veblen, and Dr. Alexandroff and Dr. Hopf, of Princeton, Professor H. M. Morse of Harvard, Professor E. W. Chittenden of Iowa, and Professor Kline of Pennsylvania. On the first day, Professor Veblen

spoke at length on the development of the subject; on the second day, Professor Morse spoke at length on critical points of functions. There were numerous shorter talks and informal discussions.

The gold medal of the Royal Astronomical Society has been awarded to Professor R. A. Semple, for his theory of the four great satellites of Jupiter.

The Royal Society of London has awarded a royal medal to Professor J. C. McLennan, for his work in spectroscopy and atomic physics.

The Messel medal of the British Society of Chemical Industry has been awarded to Dr. R. A. Millikan, of the California Institute of Technology, in recognition of his achievement in measuring the electrostatic charge of the electron.

Professor M. I. Pupin has been given the Washington award of the Western Society of Engineers, in recognition of his work in long distance telephony and radio communication.

The Deutsche Gesellschaft für Wissenschaft und Kunst of Brünn has founded an endowment in memory of Professor Emil Waelsch, of the German Technical School at Brünn, for the establishment of scholarships in geometry at that Technical School.

The University of the Northern Caucasus (Rostov on the Don) and the Polytechnic Institute of the Don, with their associated scientific societies, held a meeting on April 22, 1928, in honor of Professor D. D. Mordoukhay-Boltovskoy, to celebrate his completion of thirty years of teaching.

M. Joseph Auclair has been elected a correspondent of the Paris Academy of Sciences in the section of mechanics.

The annual William Lowell Putnam Lecture was delivered on Wednesday, March 21, 1928, at Harvard University, by Professor Constantin Carathéodory, of the University of Munich. His subject was *Selected problems of the calculus of variations.*

Professor E. W. Brown, of Yale University, has been elected an associate of the Royal Academy of Belgium.

On the occasion of its semi-centennial celebration, the University of Colorado conferred a doctorate of laws on Professor R. A. Millikan, of the California Institute of Technology.

Dr. George Sauté, of Harvard University, has been awarded one of the fellowships of the Commission for Relief in Belgium Educational Foundation, for the study of mathematics in Belgium.

The academic jubilee of Professor Ch. J. de la Vallee Poussin was celebrated on May 13, 1928, at Louvain, Belgium, under the auspices of an international committee, by fitting academic exercises and by the presentation of a bust.

It is announced that Guggenheim fellowships in astronomy, mathematics, and physics have been awarded to the following persons: Professor Perry Byerly, of the University of California, for the study of mathematical geophysics; Professor Olive C. Hazlett, of the University of Illinois, for the study of the arithmetics of linear associative algebras together with their application and interpretation in other lines of mathematics, especially the theory of numbers; Professor J. J. Hopfield, of the University of California, to study the Zeemann effect of the infra-red spectrum of oxygen and nitrogen; Dr. R. J. Kennedy, of the California Institute of Technology, for research in the theory of radiation; Professor N. C. Little, of Bowdoin College, for the study of thermo-magnetic properties of gaseous molecules; Professor F. W. Loomis, of New York University, for the study of quantum mechanics; Professor L. E. Reukema, of the University of California, for the theoretical and experimental study of electric discharge in gases at high frequencies; Professor Otto Struve, of the University of Chicago, for a theoretical study of the distribution and physical properties of diffuse matter in interstellar space; Professor W. W. Watson, of the University of Chicago, for the study of molecular spectra.

The University of Göttingen announces the following courses in mathematics and mathematical physics for its summer session of 1928 (July 9 to August 4): Professor Hilbert, Foundations of mathematics; Professor Landau, Mapping with smooth functions; Professor Prandtl, Problems of aerodynamics; Professor Courant, Development of mathematics in the 19th century; methods of analysis; Professor Betz, Theory of hydro- and aerodynamics (with demonstrations); Dr. Schuler, The top in astronomy, physics, and technology (with demonstrations); Dr. Walther, New methods in applied mathematics; Dr. Grandjot, Infinite series in number theory; Dr. Neugebauer, Mathematics in antiquity. Lectures and courses in experimental and theoretical physics will be given by Professors Pohl, Reich, Born, Franck, Kienle, Coehn, and Oldenberg, and Drs. Sponer, Rupp, and Heitler. The subjects include electrical and optical phenomena, physical optics, electrons, band spectra, astronomy, physics of the atom, photo-chemistry, and electro-acoustics. These courses are intended especially for foreign students. Further particulars may be obtained from the Secretary of the University, Wilhelmsplatz (Aula), Göttingen, Germany.

Dr. Reinhold Fürth has been promoted to an associate professorship of theoretical physics at the German University of Prague.

Associate Professor Lothar Koschmieder, of the University of Breslau, has been appointed professor of mathematics at the German Technical School at Brünn.

Dr. Wolfgang Pauli, of the University of Hamburg, has been appointed professor of theoretical physics at the Zurich Technical School.

Dr. Hans Lewy has been admitted as privat docent in mathematics at the University of Göttingen.

Professor H. Mohrmann, of Basel, has been called to the professorship of descriptive geometry at the Darmstadt Technical School.

Dr. S. Mandelbrojt has been appointed "chargé de cours" in differential and integral calculus at the University of Lille.

Professor G. A. Gibson, of the University of Glasgow, has retired.

The following have been admitted as docents in European universities: Axel Schur, for mathematics, at Bonn; B. Segre, for analytic geometry, and G. Krall, for rational mechanics, at Rome; C. Poli, for rational mechanics, at Turin; O. Nikodym at Warsaw.

The following 41 doctorates with mathematics or mathematical physics as major subject were conferred by American universities during 1927; the university, month in which the degree was conferred, minor subject (other than mathematics), and title of dissertation are given in each case if available.

R. G. Archibald, Chicago, June, *Diophantine equations in division algebras.*

W. L. Ayres, Pennsylvania, June, *Concerning continuous curves and correspondences.*

F. R. Bamforth, Chicago, December, *A classification of boundary value problems for a system of ordinary differential equations of the second order.*

L. M. Blumenthal, Johns Hopkins, June, *Lagrange resolvents in euclidean geometry.*

W. F. Cheney, Jr., Massachusetts Institute of Technology, June, physics, *Infinitesimal deformation of surfaces in riemannian space.*

Alonzo Church, Princeton, June, *Alternatives to Zermelo's assumption.*

E. U. Condon, California, December, Physics, *Theory of intensity distribution in band systems.*

T. F. Cope, Chicago, December, *An analogue of Jacobi's condition for the problem of Mayer with variable end points.*

C. C. Craig, Michigan, June, *An application of Thiele's semi-invariants to the sampling problem.*

D. R. Davis, Chicago, June, *The inverse problem of the calculus of variations in higher space.*

D. A. Flanders, Pennsylvania, June, *Double elliptic geometry in terms of point, order, and congruence.*

P. A. Fraleigh, Cornell, June, physics, *Regular bilinear transformations of sequences.*

H. H. Germond, Wisconsin, June, physics, *The effect of space charge on a three element vacuum tube.*

Lois W. Griffiths, Chicago, June, *Certain quaternary quadratic forms and diophantine equations by generalized quaternion algebras.*

Laura Guggenbühl, Bryn Mawr, June, education, *An integral equation with an associated integral condition.*

D. C. Harkin, Chicago, September, *The duality and abstract identity of the theories of modular systems and ideals.*

Ruby U. Hightower, Missouri, June, physics and astronomy, *On the classification of the elements of a ring.*

Charles Hopkins, Illinois, May, physics, *Non-abelian groups whose groups of isomorphisms are abelian.*

C. G. Jaeger, Missouri, June, physics, *A character symbol for primes relative to a cubic field.*

Vern James, Stanford, June, statistics, *Primitive linear homogeneous groups of variety two or three.*

A. R. Jerbert, University of Washington, December, physics, *Projective differential geometry of a system of linear differential equations of the first order.*

G. W. Kenrick, Massachusetts Institute of Technology, March, electrical engineering, *A new method of periodogram analysis with illustrative applications.*

James McGiffert, Columbia, May, *Particular solutions in closed form of types of linear differential equations of second order.*

E. L. Mackie, Chicago, September, *The Jacobi condition for a problem of Mayer with variable end points.*

Florence M. Mears, Cornell, June, physics, *Riesz summability for double series.*

W. M. Miller, Illinois, May, astronomy, *On the variance of interpolated observations.*

T. W. Moore, Yale, June, *The complete system of invariants of the rational space quintic curve.*

J. S. Morrell, Illinois, May, physics, *Expansion of functions in series of functions generalizing the gamma function.*

J. H. Neelley, Yale, June, *Compound singularities of the rational plane quartic curve.*

F. C. Ogg, Illinois, May, physics, *Certain configurations on cubics.*

Hillel Poritsky, Cornell, June, physics, *Topics in potential theory.*

T. H. Rawles, Yale, June, *The invariant integral and the inverse problem in the calculus of variations.*

E. H. Reimer, California, May, astronomy, *A new Eddingtonian geometry, with applications to differential geometry.*

C. H. Richardson, Michigan, June, *The statistics of sampling.*

Edward Saible, Massachusetts Institute of Technology, October, physics, *Analytic and topological transformations of closed surfaces.*

Hazel E. Schoonmaker, Cornell, June, education, *Non-monoidal involutions having a congruence of invariant conics.*

H. C. Shaub, Cornell, September, astronomy, *Rational involutorial transformations in S_4 which leave invariant ∞^4 quadratic varieties.*

A. A. Shaw, California, December, analytic mechanics, *Solution of homogeneous linear difference equations by means of infinite determinants.*

I. M. Sheffer, Harvard, June, *On the theory of linear differential equations of infinite order.*

C. H. Smiley, California, May, astronomy, *On the number of solutions of Leuschner's direct method for determining parabolic orbits.*

E. P. Starke, Columbia, July, *Certain uniform functions of rational functions.*

W. J. Trjitzinsky, California, August, celestial mechanics, *The elliptic cylinder differential equation.*

H. S. Wall, Wisconsin, June, mathematical physics, *On the Padé approximants associated with the continued fraction and series of Stieltjes.*

A. K. Waltz, Cornell, June, physics, *The steady flow of liquid through a circular hole in an infinite plane.*

G. T. Whyburn, Texas, June, organic chemistry, *Concerning continua in the plane.*

W. M. Whyburn, Texas, June, physical chemistry, *Linear boundary value problems for ordinary differential equations and their associated difference equations.*

John Williamson, Chicago, June, *Conditions for associativity of division algebras connected with non-abelian groups.*

A. S. Winsor, Johns Hopkins, June, *The composition of homologies and of homographies.*

The following graduate courses in mathematics are announced for the summer of 1928:

UNIVERSITY OF CHICAGO, first term, June 18 to July 25; second term, July 26 to August 31.—By Professor L. E. Dickson: Topics in the theory of numbers.—By Professor E. T. Bell: Theory of functions of a complex variable; Applications of analysis to the arithmetic of quadratic forms.—By

Professor E. P. Lane: Synthetic projective geometry; Projective differential geometry I.—By Professor F. D. Murnaghan: Modern hydrodynamical theory.—By Professor M. I. Logsdon: Higher plane curves.—By Professor L. M. Graves: Modern theories of integration.—By Professor R. W. Barnard: Advanced calculus; Introduction to higher algebra.—By Professor Walter Bartky: Introduction to celestial mechanics II.—By Dr. F. R. Bamforth: Differential equations.

UNIVERSITY OF COLORADO, first term, June 18 to July 21; second term, July 23, to August 24.—By Professor Light: Teacher's course in mathematics; History of mathematics; Calculus of variations.—By Professor Kempner: Advanced teacher's course in mathematics; Differential equations; Group theory and introduction to Galois theory. Second term:—By Professor Light: Statistics; Theory of equations; Calculus of variations (continued).—By Professor Kempner: Advanced teacher's course (repeated); Differential equations (continued); Group theory and introduction to Galois theory (continued).

COLUMBIA UNIVERSITY, July 9 to August 17.—By Professor W. B. Fite: Differential equations; Theory of infinite series.—By Professor J. F. Ritt: Theory of functions of a complex variable.—By Professor K. W. Lamson: Differential geometry.—By Dr. B. O. Koopman: Introduction to modern geometry.

UNIVERSITY OF ILLINOIS, June 18 to August 11.—By Professor G. A. Miller: Introduction to higher algebra, based on Bôcher; Critical study of the history of mathematics.—By Professor J. B. Shaw: Functions of a real variable.—By Professor H. R. Brahana: Projective geometry.—By Dr. L. L. Steimley: Differential equations.—By Dr. H. W. Bailey: Teachers' courses in elementary mathematics.—By Mr. G. L. Edgett: Mathematical statistics for teachers.

UNIVERSITY OF IOWA, first term, June 7 to July 20.—By Dr. M. A. Nordgaard: Subject matter and teaching of mathematics.—By Dr. Conkwright: Ordinary differential equations; Theory of functions of a complex variable.—By Professor Wylie: Elementary mechanics; Mathematics of finance; Descriptive astronomy.—By Professor Woods: Advanced coordinate geometry; Elliptic integrals.—By Professor Reilly: Algebra for high school teachers; Disability theory applied to life insurance.—By Professor Chittenden: Advanced calculus; Functions of infinitely many variables. Second term, July 23 to August 24.—By Dr. Nordgaard: The history of mathematics.—By Dr. Ward: Elementary mechanics; Theory of functions of a complex variable.—By Professor Baker: Differential equations; Geometry of forces.—By Professor Reilly: The numerical solution of equations; Frequency distributions and correlation.

UNIVERSITY OF MICHIGAN, June 25 to August 17.—By Professor W. B. Ford; Advanced calculus; Infinite series with special reference to Fourier series.—By Professor L. C. Karpinski: Teaching of algebra; History of mathematics.—By Professor Peter Field: Vector analysis.—By Professor

T. R. Running: Graphical methods.—By Professor J. W. Bradshaw: Figures of solid geometry; Selected topics in projective geometry.—By Professor T. H. Hildebrandt: Theory of functions of a complex variable; Theory of the potential.—By Professor H. C. Carver: Advanced mathematical theory of statistics.—By Professor L. A. Hopkins: Elements of mechanics.—By Professor C. J. Coe: Integral equations.—By Professor Norman Anning: Differential equations; Determinants and theory of equations.—By Professor J. A. Nyswander: Theory of probability; Finite differences.—By Mr. O. J. Peterson: Solid analytic geometry.

UNIVERSITY OF MINNESOTA, first term, June 16 to July 28; second term, July 30 to September 1. First term:—By Assistant Professor Gibbens: Differential equations; Synthetic metric geometry.—By Professor Brink: Advanced calculus; Reading in advanced mathematics. Second term:—By Professor Underhill: Reading in advanced mathematics.

UNIVERSITY OF NEBRASKA, June 4 to July 13.—By Professor Brenke: Differential equations.—By Professor Gaba: Advanced euclidean geometry; Foundations of algebra and geometry.—By Professor Camp: Advanced algebra.—By Professor Pierce: Theory of equations.—By Mr. Harper: Elements of the mathematical theory of statistics.

OHIO STATE UNIVERSITY, June 18 to August 31.—By Professor C. L. Arnold: The teaching of mathematics; Differential equations.—By Professor C. C. Morris: Theory of equations; Mathematical statistics.—By Professor J. H. Weaver: Differential geometry; Introduction to higher algebra.

UNIVERSITY OF PENNSYLVANIA, July 2 to August 11.—By Professor H. H. Mitchell: Probability.—By Professor M. J. Babb: College geometry.—By Professor Arnold Dresden, of Swarthmore College: Elliptic functions; Differential equations.—By Professor J. R. Kline: Theories of integration.—By Professor J. M. Thomas: Introduction to tensor analysis.

STANFORD UNIVERSITY, June 21 to August 31.—By Dr. Alfred Errera (Belgium): Topics selected from the foundations of mathematics; Geometry and the elements of axiomatics.—By Professor E. Hille (Princeton): Differential equations; Theory of functions of a complex variable.

UNIVERSITY OF TEXAS, first term, June 5 to July 16.—By Professor R. L. Moore: Functions of real variables; Foundations of geometry.—By Professor E. L. Dodd: Infinite processes; Mathematical statistics.—By Professor H. J. Ettlinger: Boundary value problems; Ruler and compass constructions.—By Professor C. D. Rice: Differential geometry; Advanced calculus.—By Professor P. M. Batchelder: Teaching problems in mathematics.—By Professor C. M. Cleveland: Advanced calculus.—By Professor Goldie Horton: Theory of equations. Second term, July 16 to August 27.—By Professor H. J. Ettlinger: Boundary value problems; Definite integrals.—By Professor P. M. Batchelder: Teaching problems in mathematics.—By Professor C. M. Cleveland: Advanced calculus.—By Professor R. G. Lubben: Non-euclidean geometry.—By Professor G. T. Whyburn: Functions of real variables.

UNIVERSITY OF WISCONSIN, six weeks session, June 25 to August 3.—By Professor J. H. Taylor: Differential equations; Differential geometry.—By Professor E. B. Van Vleck: Modern geometrical concepts; The location of the roots of an algebraic equation. Special nine weeks course for graduates, June 25 to August 24.—By Professor R. E. Langer: Partial differential equations; Calculus of variations.—By Professor Warren Weaver: Theory of potential; Introduction to statistical mechanics.

Dr. Ethel L. Anderton, of Smith College, has been appointed assistant professor of mathematics at Mount Holyoke College.

Professor Léon Brillouin, of the Collège de France, has been appointed acting professor of theoretical physics at the University of Wisconsin, for the academic year 1927–28.

Mr. Alexander Cook, lecturer at the University of Alberta, has been promoted to an assistant professorship of mathematics.

Dr. H. N. Davis, professor of mechanical engineering at Harvard University, has been elected president of the Stevens Institute of Technology.

Assistant Professor Oystein Ore, of Yale University, has been made associate professor of mathematics.

Lieutenant Colonel A. J. C. Cunningham, known for his work in number theory and the preparation of factor tables, is dead. Colonel Cunningham had been a member of the American Mathematical Society since 1905.

Professor Krishna Prasad Dé, of University College, Rangoon, is dead. Professor Dé was a member of the American Mathematical Society.

Professor R. W. Genese, of the University College of Wales, Aberystwyth, died January 21, 1928, at the age of seventy-nine. Professor Genese was a member of the American Mathematical Society.

Professor J. L. Heiberg, of Copenhagen, known for his work in the history of Greek mathematics, died January 4, 1928, at the age of seventy-two.

Professor Hendrik Antoon Lorentz, of the University of Leyden, the distinguished physicist, died February 4, 1928, at the age of seventy-four.

Dr. W. C. Eglin, vice-president and chief engineer of the Philadelphia Electric Company and president of the Franklin Institute, died February 7, 1928.

William Wallace Payne, of the observatory of the National Watch Company, Elgin, Ill., the founder of Popular Astronomy, died January 29, 1928, at the age of ninety-one.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

BALL (W. W. R.). Histoire des mathématiques. Paris, Hermann, 1927. 336 pp.

BARBARIN (P.). La géométrie non-euclidienne. 3e édition suivie de notes sur la géométrie non-euclidienne dans ses rapports avec la physique mathématique par A. Buhl. (Scientia, No. 15.) Paris, Gauthier-Villars, 1928. 176 pp.

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BOHDE (H.). Mathematisch-technische Zahlentafeln. 5te vermehrte Auflage. Berlin, Springer, 1927.

BUHL (A.). See BARBARIN (P.).

CASSINIS (G.). Calcolo numerici, grafici e meccanici. Pisa, Mariotti-Paccini, 1928. 6+672 pp.

COHN-VOSSEN (S.). See KLEIN (F.).

COURANT (R.). See KLEIN (F.).

DE DONDER (T.). Théorie des invariants intégraux. Paris, Gauthier-Villars, 1927.

DUARTE (F. J.). Nouvelles tables de log $n!$ à 33 décimales depuis $n=1$ jusqu'à $n=3000$. Genève, Albert Kundig, et Paris, Index Generalis, 1927. 24+136 pp.

EISENHART (L. P.). Non-Riemannian geometry. (American Mathematical Society Colloquium Publications, volume VIII.) New York, American Mathematical Society, 1927. 8+184 pp.

ENCYKLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN. 2ter Band: Analysis. 3ter Teil, 2te Hälfte. Leipzig, Teubner, 1923–27. 14 + 675+1648 pp.

ENCYKLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN. 3ter Band: Geometrie. 3ter Teil. Leipzig, Teubner, 1902–27. 606+206 pp.

ENRIQUES (F.). L'évolution des idées géométriques dans la pensée grecque. Point, ligne, surface. Traduit sur la 3e édition italienne par Maurice Solovine. Paris, Gauthier-Villars, 1927. 48 pp.

GILLAIN (O.). La science égyptienne. L'arithmétique au moyen empire. Bruxelles, Edition de la Fondation Egyptologique Reine Elisabeth, 1927. 16+326 pp.

GODFREY (C.) and SIDDONS (A. W.). Four-figure tables. Cambridge, University Press, 1927. 40 pp.

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KNOPP (K.). Aufgabensammlung zur Funktionentheorie. Teil 2: Aufgaben zur höheren Funktionentheorie. (Sammlung Göschen.) Berlin, de Gruyter, 1928. 143 pp.

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YOUNGSON (P.) and BENNETT (T. A.). Practical mathematics for marine engineers: First class. Glasgow, Munro, 1927. 571 pp.

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1. Typewrite words and the very simplest formulas only.
2. *Do not* try to typewrite any complex formulas. Write them.
3. Keep a copy, and send the editors two copies, if you can.
4. *Do not* underline any symbols or any formulas.
5. Underline theorems with blue pencil (avoid ink).
6. Follow our recent styles in abbreviations, footnotes, etc.
7. Write carefully the (often misunderstood) capitals C K P S V W X Z.
8. Write ϵ, not ε. Write very carefully $\gamma\ \eta\ \kappa\ \lambda\ \nu\ \tau\ \upsilon\ \chi\ \omega$.
9. Among Greek capitals, use only $\Gamma\ \Delta\ \Theta\ \Lambda\ \Xi\ \Pi\ \Sigma\ \Phi\ \Psi\ \Omega$.
10. Punctuate carefully, especially in formulas; thus: $1, 2, \cdots, n$.
11. Use the solidus (/) to avoid fractions in solid lines.
12. Use fractional exponents to avoid root signs everywhere.
13. Use extra symbols to avoid complicated exponents.
14. In typewritten formulas, 1 means "one"; to indicate "ell" in formulas, backspace and overprint /; thus: l. Similarly, O means "zero"; to indicate "cap O," backspace and overprint period; thus: O.
15. Avoid a dash over a letter, except for those shown below.
16. Some samples of unusual types available on monotype machines follow. A more complete list of all such types will be sent on request.

Light Face Greek—$\alpha\ \beta\ \gamma\ \cdots$ (all) A B $\Gamma\ \cdots$ (all).

★ Light Greek Superiors—$^{\Delta}$ and $^{\alpha\ \beta\ \gamma\ \cdots}$ (all except $^{\iota}$ and ^{o}).

★ Light Greek Inferiors—$_{\Delta\ \Lambda\ \Sigma\ \Omega}$ and $_{\alpha\ \beta\ \gamma\ \cdots}$ (all except $_{\iota}$ and $_{o}$).

* Boldface Greek—$\boldsymbol{\alpha\ \beta\ \delta\ \varepsilon\ \zeta\ \eta\ \theta\ \mu\ \nu\ \xi\ \pi\ \rho\ \sigma\ \omega}$ and $\boldsymbol{\Omega}$.

* Lightface German—$\mathfrak{a\ b\ c\ d\ p\ q}\ \mathfrak{A\ B\ C\ D\ E\ F\ G\ H\ J\ L\ M\ P\ Q\ R\ S\ T\ U\ V\ W\ X\ Y\ Z}$.

* Boldface German—$\mathfrak{d\ A\ B\ D}$

Script (special font $\mathcal{A\ B\ C}\ \cdots$ (all). No lower case manufactured.

* Hebrew— בכ אא troublesome to handle.

★ Dashed Italics—$\overline{A}\ \overline{a}\ \overline{B}\ \overline{b}\ \overline{C}\ \overline{c}\ \overline{E}\ \overline{e}\ \overline{F}\ \overline{f}\ \overline{G}\ \overline{g}\ \overline{H}\ \overline{I}\ \overline{i}\ \overline{K}\ \overline{k}\ \overline{M}\ \overline{m}\ \overline{N}\ \overline{n}\ \overline{P}\ \overline{p}\ \overline{q}\ \overline{R}\ \overline{r}\ \overline{s}\ \overline{t}\ \overline{u}\ \overline{v}\ \overline{X}\ \overline{x}\ \overline{Y}\ \overline{y}\ \overline{Z}\ \overline{z}$

★ Tilda Italics—$\tilde{A}\ \tilde{a}\ \tilde{e}\ \tilde{N}\ \tilde{n}\ \tilde{O}\ \tilde{r}\ \tilde{u}\ \tilde{y}$

★ Tilda Greek—$\tilde{\alpha}\ \tilde{\epsilon}\ \tilde{\eta}\ \tilde{\omega}\ \tilde{\varpi}$

★ Dashed Greek—$\overline{\alpha}\ \overline{\beta}\ \overline{\gamma}\ \overline{\delta}\ \overline{\eta}\ \overline{\theta}\ \overline{\mu}\ \overline{\nu}\ \overline{\rho}\ \overline{\omega}\ \overline{\Gamma}$

★ Dotted Italic—$\dot{a}\ \ddot{a}\ \dot{e}\ \ddot{e}\ \ddot{g}\ \ddot{\imath}\ \dot{m}\ \dot{n}\ \dot{q}\ \dot{r}\ \ddot{r}\ \dot{u}\ \ddot{u}\ \dot{x}\ \ddot{x}\ \dot{y}\ \ddot{y}\ \dot{z}\ \ddot{z}$.

★ Dotted Greek—$\dot{\eta}\ \ddot{\eta}\ \dot{\theta}\ \ddot{\theta}\ \dot{\xi}\ \dot{\psi}\ \ddot{\psi}\ \ddot{\omega}$ (single dotted $\zeta\ \phi\ \delta\ \beta\ \gamma$; double dotted γ readily available).

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Meetings of the Society have been fixed at the following times and places:

REED COLLEGE, Portland, Oregon, June 2, 1928. (The San Francisco Section.)

Abstracts must be in the hands of the Secretary of the Section, Professor B. A. Bernstein, Berkeley, California, not later than May 12.

AMHERST COLLEGE, SUMMER MEETING AND COLLOQUIUM, September 4-7, 1928.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than August 3. A series of colloquium lectures will be delivered by Professor A. B. Coble on *The determination of the tritangent planes of the space sextic of genus four.*

NEW YORK CITY, October 27, 1928.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th St., New York City, not later than October 6.

BERKELEY, Calfornia, October 27, 1928. (The San Francisco Section.)

Abstracts must be in the hands of the Secretary of the Section, Professor B. A. Bernstein, Berkeley, California, not later than October 6. Professor E. T. Bell will give an address on *The theory of algebraic numbers in the light of Kronecker's program.*

LAWRENCE, KANSAS, December 1, 1928. (The Southwestern Section.)

Abstracts must be in the hands of the Secretary of the Section, Professor E. B. Stouffer, Lawrence, Kansas, not later than November 10.

NEW YORK CITY, ANNUAL MEETING, December 27-28, 1928.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th Street, New York City, not later than November 29.

R. G. D. RICHARDSON,
Secretary of the Society.

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

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The initiation fees ($5.00) and the annual dues ($6.00) of members of the Society are payable to the Treasurer of the Society, W. B. FITE, 501 West 116th Street, New York.

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